# Free Surface Hydrodynamics
# 2DH and 3D Shallow Water Equations

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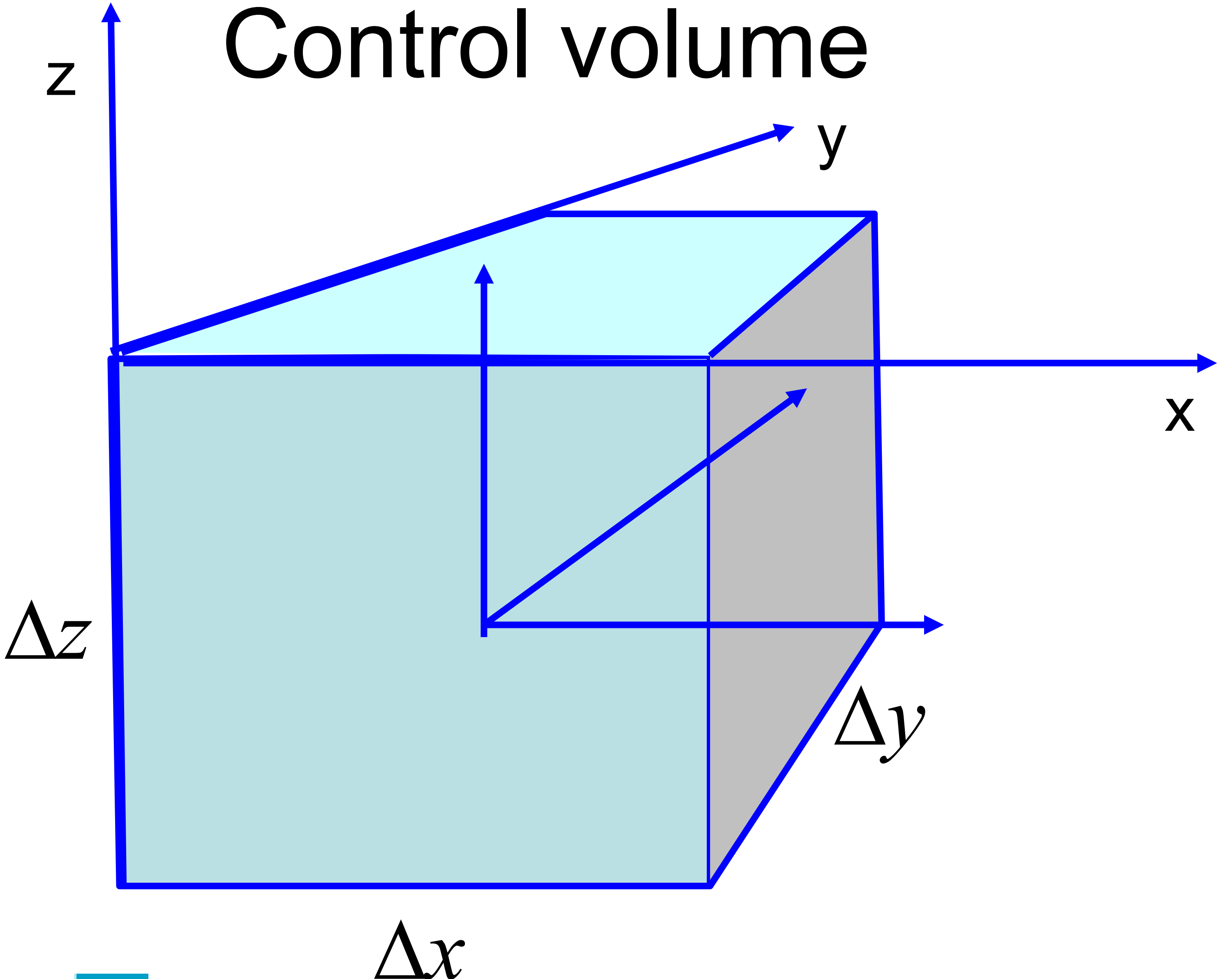
# Contents

- Main assumptions and derivation from Navier-Stokes Equations
- Some simple limit cases
- (A bit on) numerical models
- Typical applications

# Control volume

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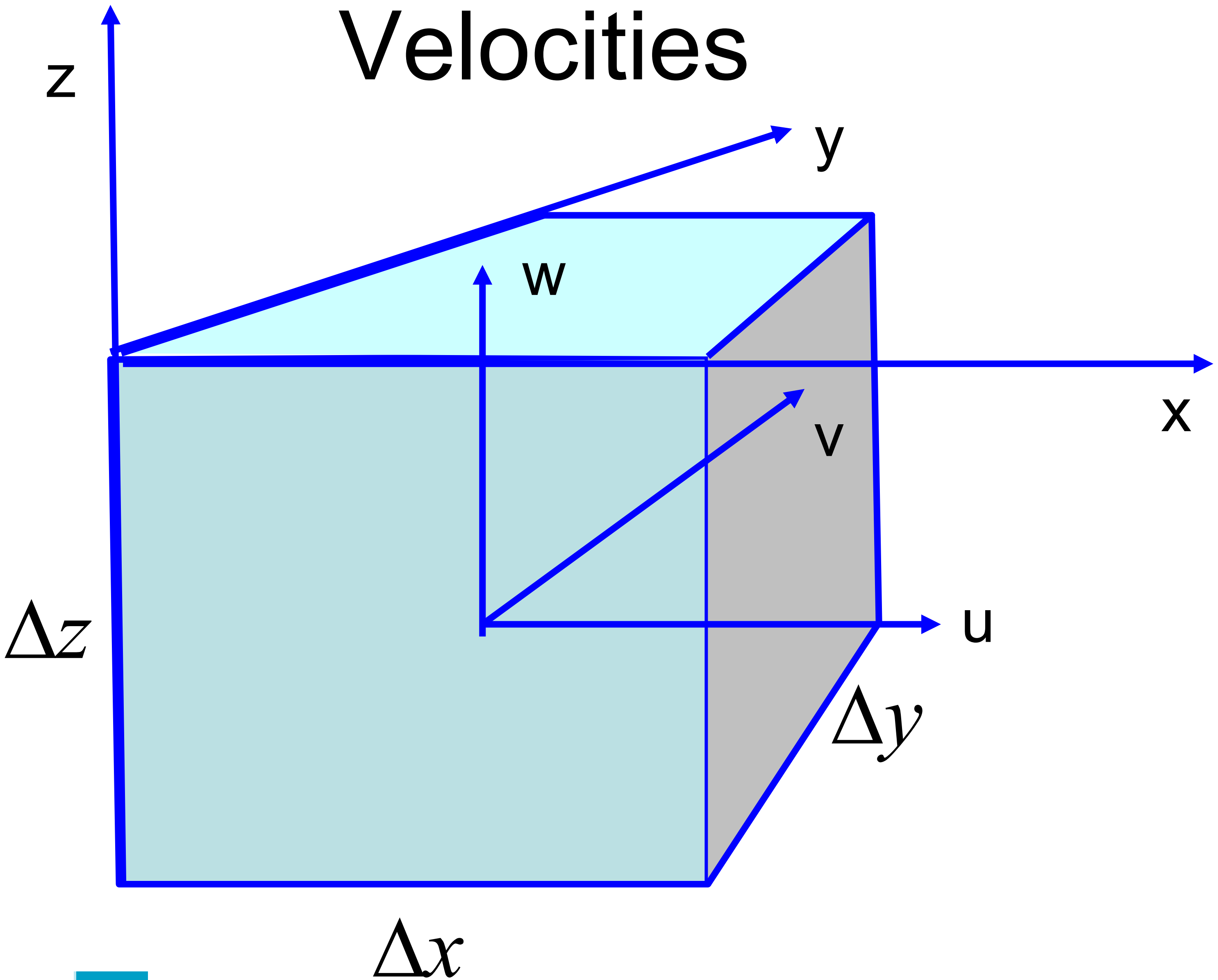
# Velocities

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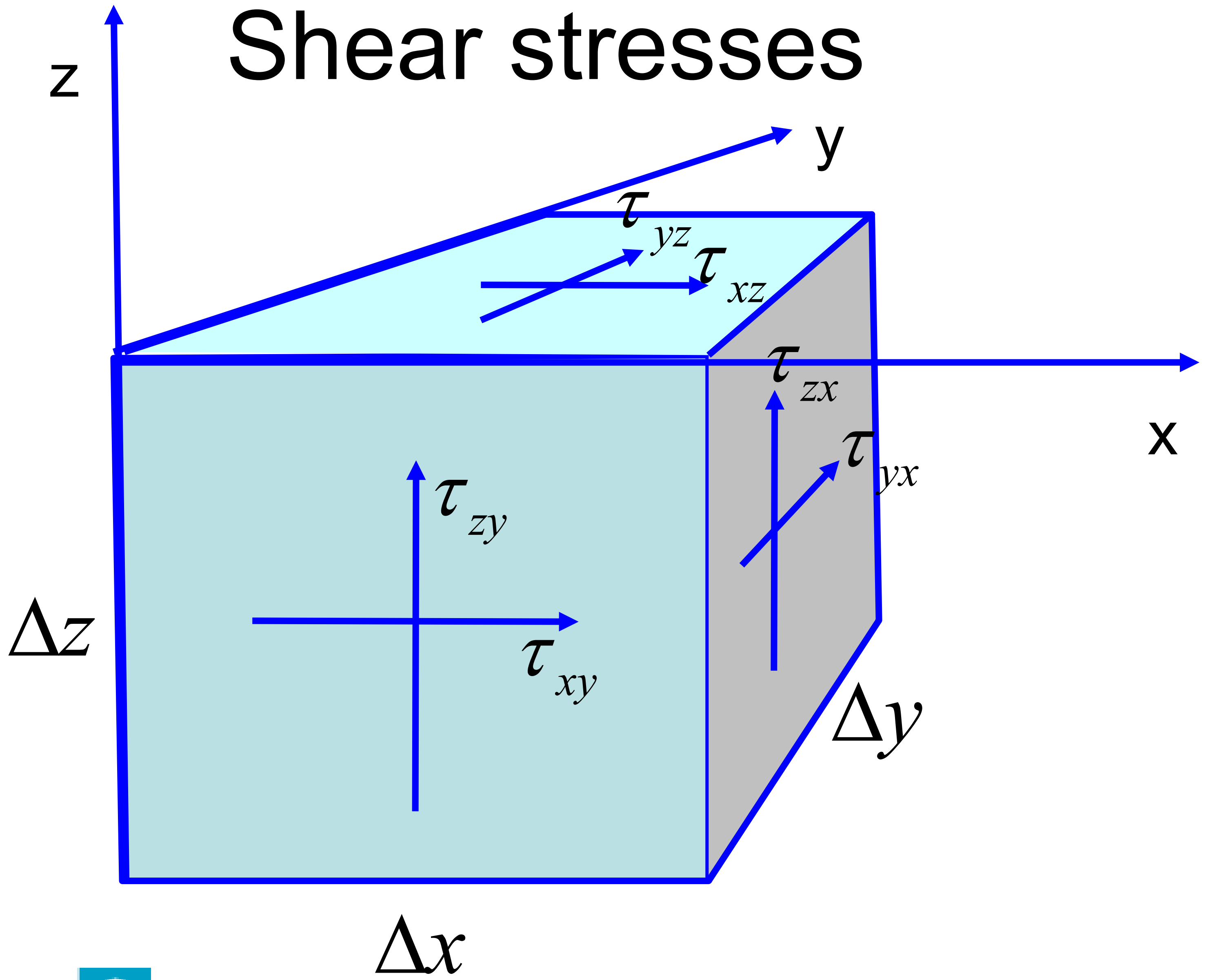
# Shear stresses

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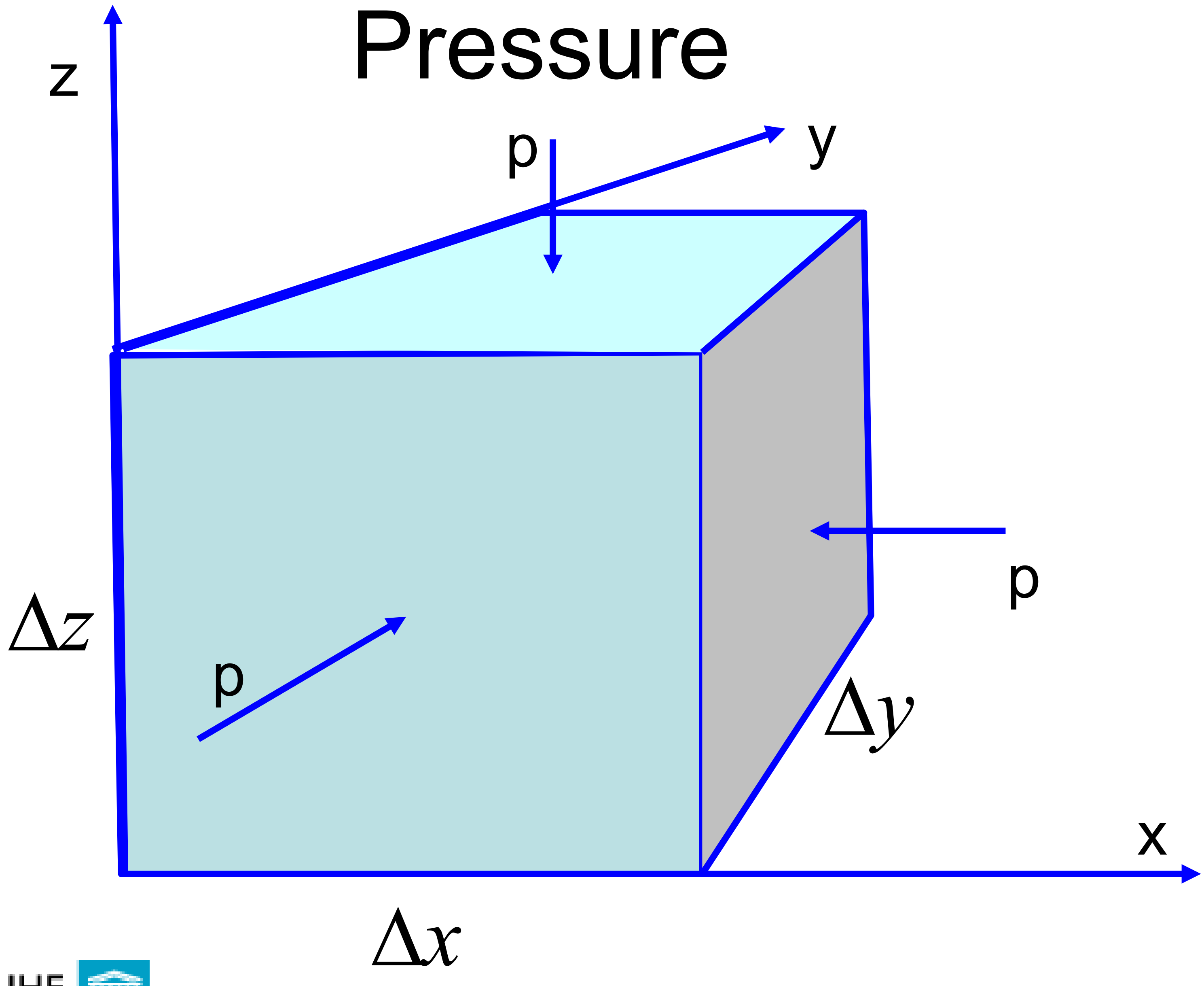
# Pressure

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# Momentum balance

$$\frac{du}{dt} - f_{cor}v = \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} - \frac{\partial p}{\partial x}\right)$$

$$\frac{dv}{dt} + f_{cor}u = \frac{1}{\rho}\left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} - \frac{\partial p}{\partial y}\right)$$

$$\frac{dw}{dt} = \frac{1}{\rho}\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} - \frac{\partial p}{\partial z}\right) - g$$

# Mass balance

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

# Assumption 1: incompressible flow

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

# Averaging momentum balance over short timescales

- Turbulence
  - Reynolds stresses
  - Approximated by turbulent shear stresses

$$\tau_{xy} = \rho \nu_h \frac{\partial u}{\partial y}, \ \tau_{yx} = \rho \nu_h \frac{\partial v}{\partial x}, \ \tau_{xz} = \rho \nu_v \frac{\partial u}{\partial z}, \ \tau_{yz} = \rho \nu_v \frac{\partial v}{\partial z}$$

$$\tau_{zx} = \rho \nu_h \frac{\partial w}{\partial x}, \tau_{zy} = \rho \nu_h \frac{\partial w}{\partial y}$$

# Shallow water approximation

- Horizontal scales >> vertical scales
- Vertical velocities << horizontal velocities
- Neglect vertical acceleration

$$\frac{dw}{dt} = \frac{1}{\rho}\left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} - \frac{\partial p}{\partial z}\right) - g \Rightarrow$$

$$\frac{\partial p}{\partial z} = -\rho g$$

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# Hydrostatic pressure

- Inhomogeneous (density not constant):

$$p = p_a + g\int_z^{\eta} \rho dz$$

- Homogeneous (density constant):

$$p = p_a + \rho g(\eta - z)$$

# Shallow Water Equations (3D)

$$\frac{du}{dt} - f_{cor} v = \frac{\partial}{\partial y}\left(\nu_h \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\nu_v \frac{\partial u}{\partial z}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{w_x}{\rho}$$

$$\frac{dv}{dt} + f_{cor} u = \frac{\partial}{\partial x}\left(\nu_h \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu_v \frac{\partial v}{\partial z}\right) - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{w_y}{\rho}$$

Acceleration

Coriolis

Horizontal diffusion

Vertical diffusion

Horizontal pressure gradient

Wave forcing

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# Boundary conditions

$$w = 0$$

$$w = \frac{\partial \eta}{\partial t}$$

$$\rho \nu_v \frac{\partial u}{\partial z} = \tau_{bx}$$

$$\rho \nu_v \frac{\partial u}{\partial z} = \tau_{sx}$$

$$\rho \nu_v \frac{\partial v}{\partial z} = \tau_{by}$$

$$\rho \nu_v \frac{\partial v}{\partial z} = \tau_{sy}$$

Bottom (z=-d)

Surface ($z = \eta$)

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# Depth-averaged mass balance

$$\int_{-d}^{\eta} \frac{\partial u}{\partial x} dz + \int_{-d}^{\eta} \frac{\partial v}{\partial y} dz + \int_{-d}^{\eta} \frac{\partial w}{\partial z} dz = 0 \Rightarrow$$

$$\Rightarrow \frac{\partial h\overline{u}}{\partial x} + \frac{\partial h\overline{v}}{\partial y} + w(\eta) - w(-h) = 0 \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial h\overline{u}}{\partial x} + \frac{\partial h\overline{v}}{\partial y} + \frac{\partial \eta}{\partial t} = 0}$$

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# From moving to fixed frame of reference

$$u = f(t, x, y, z)$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

# From moving to fixed frame of reference

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

Inertia

Advection

# Shallow Water Equations (3D)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-f_{cor}v=\frac{\partial}{\partial y}\left(\nu_h\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial z}\left(\nu_v\frac{\partial u}{\partial z}\right)-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{w_x}{\rho}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+f_{cor}u=\frac{\partial}{\partial x}\left(\nu_h\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\nu_v\frac{\partial v}{\partial z}\right)-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{w_y}{\rho}$$

$$\frac{\partial h\overline{u}}{\partial x}+\frac{\partial h\overline{v}}{\partial x}+\frac{\partial \eta}{\partial t}=0$$

$$p=p_a+\int_z^{\eta}\rho g dz$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

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# Depth-averaged momentum balance

$$\frac{\partial \overline{u}}{\partial t} + \overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} - f_{cor}\overline{v} = \frac{\partial}{\partial y} D_h \frac{\partial \overline{u}}{\partial y} + \frac{\tau_{sx}}{\rho h} - \frac{\tau_{bx}}{\rho h} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} - g\frac{\partial \eta}{\partial x} + \frac{F_x}{\rho h}$$

$$\frac{\partial \overline{v}}{\partial t} + \overline{u}\frac{\partial \overline{v}}{\partial x} + \overline{v}\frac{\partial \overline{v}}{\partial y} + f_{cor}\overline{u} = \frac{\partial}{\partial x} D_h \frac{\partial \overline{v}}{\partial x} + \frac{\tau_{sy}}{\rho h} - \frac{\tau_{by}}{\rho h} - \frac{1}{\rho}\frac{\partial p_a}{\partial y} - g\frac{\partial \eta}{\partial y} + \frac{F_y}{\rho h}$$

Acceleration

Advection

Coriolis

Horizontal diffusion

Wind shear stress

Bed shear stress

Atmospheric pressure

Water level gradient

Wave forcing

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# Free Surface Hydrodynamics
# 2DH and 3D Shallow Water Equations
# Simple limit cases

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# Contents

- Stationary, uniform flow
- 1D tidal wave
- 1D St Venant equations
- Backwater curve
- Stationary wind setup
- Vertical profile of uniform, stationary flow

# Limit case: stationary, uniform flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f_{cor}v = \frac{\partial}{\partial y}D_h\frac{\partial u}{\partial y} + \frac{\tau_{sx}}{\rho h} - \frac{\tau_{bx}}{\rho h} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} - g\frac{\partial \eta}{\partial x} + \frac{F_x}{\rho h}$$

$$\Rightarrow \frac{\tau_{bx}}{\rho h} = -g\frac{\partial \eta}{\partial x} \Rightarrow \tau_{bx} = -\rho g h\frac{\partial \eta}{\partial x}$$

Question: given Chezy law, how can you compute velocity u?

# Chezy law

$$\tau_{bx} = -\rho g h \frac{\partial \eta}{\partial x}$$

$$\tau_{bx} = \rho g \frac{|u|u}{C^2}$$

$$\rho g \frac{|u|u}{C^2} = -\rho g h \frac{\partial \eta}{\partial x} \Rightarrow |u|u = -C^2 h \frac{\partial \eta}{\partial x} = C^2 h S$$

$$\boxed{u = C\sqrt{hS}\ ,\ \ S > 0}$$

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# Limit case: 1D tidal wave

- Very long tidal wave in deep channel

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f_{cor}v = \frac{\partial}{\partial y} D_h \frac{\partial u}{\partial y} + \frac{\tau_{sx}}{\rho h} - \frac{\tau_{bx}}{\rho h} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} - g\frac{\partial \eta}{\partial x} + \frac{F_x}{\rho h}$$

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} = 0$$

From continuity eq.

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# Limit case: 1D tidal wave

$$\frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} = 0$$

$$\eta = \widehat{\eta}\cos(\omega t - kx)$$

$$u = \widehat{u}_1 \cos(\omega t - kx)$$

$$\omega = \frac{2\pi}{T},\ k = \frac{2\pi}{L}$$

$$\frac{\partial \eta}{\partial t} = -\omega\widehat{\eta}\sin(\omega t - kx)$$

$$\frac{\partial u}{\partial t} = -\omega\widehat{u}\sin(\omega t - kx)$$

$$\frac{\partial \eta}{\partial x} = k\widehat{\eta}\sin(\omega t - kx)$$

$$\frac{\partial u}{\partial x} = k\widehat{u}\sin(\omega t - kx)$$

$$-\omega\widehat{u} + gk\widehat{\eta} = 0$$

$$-\omega\widehat{\eta} + hk\widehat{u} = 0$$

# Shallow water wave celerity

- After substituting u in second equation:

$$\omega / k = \boxed{c = \sqrt{gh}} \qquad \omega = \frac{2\pi}{T}, \; k = \frac{2\pi}{L}$$

- First equation:

$$u = \frac{c}{h}\widehat{\eta} = \sqrt{\frac{g}{h}}\widehat{\eta}\cos(\omega t - kx)$$

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# How to use it

- Period T is given (approx. 12 hrs)
- Celerity c depends only on water depth
- Velocity u depends on water depth and tidal amplitude
- Example: given water depth of 20 m, tidal amplitude of 1 m, estimate celerity and amplitude of velocity

# Limit case: 1D St Venant equations

- Neglect v velocity and all gradients with y

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f_{cor}v = \frac{\partial}{\partial y}D_h\frac{\partial u}{\partial y} + \frac{\tau_{sx}}{\rho h} - \frac{\tau_{bx}}{\rho h} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} - g\frac{\partial \eta}{\partial x} + \frac{F_x}{\rho h}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = 0$$

# Limit case: backwater curve

- St Venant + stationary: neglect d/dt

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f_{cor}v = \frac{\partial}{\partial y}D_h\frac{\partial u}{\partial y} + \frac{\tau_{sx}}{\rho h} - \frac{\tau_{bx}}{\rho h} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} - g\frac{\partial \eta}{\partial x} + \frac{F_x}{\rho h}$$

$$\frac{\partial \frac{1}{2}u^2}{\partial x} = -\frac{\tau_{bx}}{\rho h} - g\frac{\partial h}{\partial x} - g\frac{\partial z_b}{\partial x} \qquad \eta = z_b + h$$

$$\frac{\partial\left(h + \frac{u^2}{2g}\right)}{\partial x} = -\frac{\tau_{bx}}{\rho g h} - \frac{\partial z_b}{\partial x}$$

# Limit case: stationary wind setup

- Wind exerts surface shear stress
- If there is a closed boundary , the cross-shore velocity goes to zero
- Wind stress term is compensated by surface slope term

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f_{cor}v = \frac{\partial}{\partial y}D_h\frac{\partial u}{\partial y} + \frac{\tau_{sx}}{\rho h} - \frac{\tau_{bx}}{\rho h} - \frac{1}{\rho}\frac{\partial p_a}{\partial x} - g\frac{\partial \eta}{\partial x} + \frac{F_x}{\rho h}$$

$$\frac{\partial \eta}{\partial x} = \frac{\tau_{sx}}{\rho g h} \Rightarrow \Delta\eta = \frac{\tau_{sx}}{\rho g h}\Delta x$$

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# Setup question

- Wind shear stress is 1 N/m2
- Length of sea or lake is 100 km
- Water depth is 10 m
- How big is water level difference
- Is it different for a lake or a sea?

# 3D limit case: vertical profile of uniform, stationary flow

- Shear stress term balances pressure gradient term

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+f_{cor}u=\frac{\partial}{\partial x}\left(\nu_h\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\nu_v\frac{\partial v}{\partial z}\right)-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{w_y}{\rho}$$

- Pressure gradient given by surface slope term:

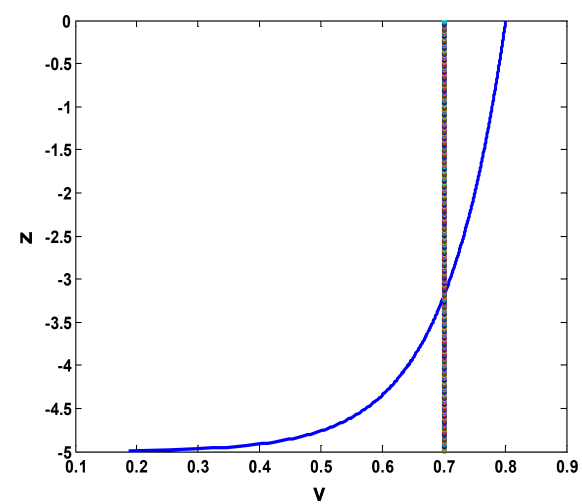
$$\frac{\partial}{\partial z}\left(\nu_v\frac{\partial v}{\partial z}\right)=g\frac{\partial \eta}{\partial y}$$

- Parabolic viscosity distribution

$$\nu_v=-\kappa v_* z\frac{(h+z)}{h}$$

- Solution: logarithmic profile:

$$v=\frac{v_*}{\kappa}\ln\frac{h+z}{z_0}$$

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(Derivation in lecture notes)

# Why these analyses if you have numerical models?

- Numerical models can be wrong
- Need to understand the outcome
- Need to be able to check at least the order of magnitude of the outcome