About Module 2 – Hydraulics and Hydrology

- Free surface hydrodynamics (35%)
- Engineering hydrology (35%)
- GIS and remote sensing (30%)
Part-1: Fundamental principles and equations

- Continuity and Momentum principles
- Euler and Navier-Stokes equations
- Bernoulli equation and applications

Continuity principle

- Expresses the conservation of mass in a control volume occupied by a fluid.
- Obtained by equating the change in fluid mass to the difference between the rate of mass IN and OUT in a given time.
- Change in mass in time $\delta t$ is

$$
(\rho - (\rho + \frac{\partial \rho}{\partial t})) \delta x \delta y \delta z = - \frac{\partial \rho}{\partial t} \delta t \delta x \delta y \delta z
$$

- Change in mass due to the change in $\rho$ and $u$ (in $x$-direction)

$$
(\rho u A_x + A_x \frac{\partial \rho u}{\partial x} \delta x - \rho u A_x) \delta t = A_x \delta x \frac{\partial \rho u}{\partial x} \delta t = \frac{\partial \rho u}{\partial x} \delta t \delta x \delta y \delta z
$$
Continuity principle ...

- General form of continuity equation:

\[
\frac{\partial \rho}{\partial t} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} = 0
\]

- For incompressible fluid with constant \( \rho \):

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

- For a steady and incompressible flow in a pipe:

\[ Q = V_1 A_1 = V_2 A_2 \]

Also valid for a steady continuous flow in an open channel.

Continuity principle ...

- For unsteady flow in an open channel:

\[
\frac{\partial Q}{\partial x} + B \frac{\partial h}{\partial t} = 0 \quad \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0
\]
Momentum principle

- Inertia force: To change an existing motion of a body of mass $M$, it is necessary to apply a force $F$ to this mass, which causes an acceleration $a = \frac{dV}{dt}$, such that

$$F = M \frac{dV}{dt}$$

- This is the Newton's Equation of Motion. The product $M\frac{dV}{dt}$ is the inertia force.

Two types of inertia forces are considered

- Due to “local acceleration” → change in velocity in time $\frac{dV}{dt}$
- Due to “convective acceleration” → change in velocity over a distance $\nu \frac{dV}{dx}$

Momentum principle …

- Inertia force components in 3 dimensions ($x, y, z$ axes):

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$$

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)$$

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right)$$
Momentum principle ...

Applied forces:
- Gravity force \( F_g = Mg \)
- Pressure force

\[
p(\frac{\partial^2 \Phi}{\partial y \partial z}) - \left( p + \frac{\partial p}{\partial x} \right) \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial x} = -\frac{\partial p}{\partial x} \frac{\partial \Phi}{\partial y} \frac{\partial \Phi}{\partial z}
\]

- Viscous force (recall Newton’s Law of Viscosity)

\[
\tau = \mu \frac{du}{dy}
\]

Pressure forces acting on two opposite faces on the yz-plane.

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Dynamic Viscosity ( \mu ) (N s/m²) ( \times 10^3 )</th>
<th>Kinematic Viscosity ( \nu ) (m²/s) ( \times 10^{-6} )</th>
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</tr>
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</table>

Shear stress

Coefficient of viscosity

Momentum principle ...

In 3 dimensions

\[
\frac{\partial \sigma_{xx}}{\partial x} \frac{\partial \Phi}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \frac{\partial \Phi}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \frac{\partial \Phi}{\partial z} + \frac{\partial \sigma_{yy}}{\partial y} \frac{\partial \Phi}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \frac{\partial \Phi}{\partial z} + \frac{\partial \sigma_{zz}}{\partial z} \frac{\partial \Phi}{\partial z} = \frac{\partial p}{\partial x} \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial x} - \left( p + \frac{\partial p}{\partial x} \right) \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial x}
\]

Convention used to represent shear stresses on various planes. The first subscript represents the axis normal to the plane and the second subscript represents the axis along which the force is acting.
Momentum equation

- Obtained from balancing inertia and applied forces

- Euler equation neglects viscous forces

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g
\]

Convention used to represent shear stresses on various planes. The first subscript represents the axis normal to the plane and the second subscript represents the axis along which the force is acting.

Momentum equation ...

- Navier-Stokes equation also includes viscous forces

\[
\frac{du}{dt} = \frac{1}{\rho} \left( -\frac{\partial p}{\partial x} + \frac{\partial \tau_{x\alpha}}{\partial x} + \frac{\partial \tau_{\alpha x}}{\partial y} + \frac{\partial \tau_{\alpha y}}{\partial z} \right) \\
\frac{dv}{dt} = \frac{1}{\rho} \left( -\frac{\partial p}{\partial y} + \frac{\partial \tau_{y\alpha}}{\partial x} + \frac{\partial \tau_{x\alpha}}{\partial y} + \frac{\partial \tau_{\alpha y}}{\partial z} \right) \\
\frac{dw}{dt} = \frac{1}{\rho} \left( -\frac{\partial p}{\partial z} + \frac{\partial \tau_{z\alpha}}{\partial x} + \frac{\partial \tau_{x\alpha}}{\partial y} + \frac{\partial \tau_{\alpha y}}{\partial z} \right) - g
\]

Convention used to represent shear stresses on various planes. The first subscript represents the axis normal to the plane and the second subscript represents the axis along which the force is acting.
Bernoulli equation

- Can be derived from Euler equation
- Steady flow assumption is used
- Has wide application in hydraulics
- Represents the total energy and signifies that total the energy remains constant

\[ \frac{V^2}{2g} + \frac{p}{\rho g} + z = \text{Constant} \]

Part-2: 1D Channel Flow

- Steady-uniform flow
- Friction coefficient and velocity distribution
- Specific discharge, critical depth
- Non-uniform gradually varied flow (backwater curve)
- Unsteady flow
  - St-Venant equation
  - Kinematic wave
Steady-uniform flow

- Velocity is constant (i.e. $\partial V/\partial t = 0$) and is uniform in space (i.e. in one dimension $\partial V/\partial x = 0$).
- The water surface line remains parallel to the channel bed profile.
- Gravity force to be balanced by the shear force due to the bed resistance.

$$\rho g \delta x A \sin \theta = \delta x \int_0^P \tau_b \, dP \quad \Rightarrow \quad \tau_b = \rho g \sin \theta \frac{A}{P} = \rho g R \sin \theta$$

Remember:
- $P$ is wetted perimeter
- Hydraulic radius, $R = A / P$

For small slope

$$\sin \theta \approx \tan \theta = \frac{\partial z_b}{\partial x} = S_0 \quad \tau_b = \rho g RS_0$$

For turbulent flows, the shear stress can be assumed to be related to the average velocity $V$ and friction coefficient $C_f$ as

$$\tau_b = \rho C_f V^2$$

Finally

$$V = \sqrt{\frac{g}{C_f} RS_0}$$
Steady-uniform flow …

- Chezy’s formula
  \[ V = C \sqrt{RS_0} \]
  \[ Q = CA \sqrt{RS_0} = C \frac{A^{3/2}}{P^{1/2}} S_0^{1/2} \]

- Manning’s formula
  \[ V = \frac{1}{n} R^{3/2} S_0^{1/2} \]
  \[ Q = \frac{1}{n} A R^{3/2} S_0^{1/2} = \frac{1}{n} \frac{A^{3/2}}{P^{1/2}} S_0^{1/2} \]

Remember:
- Discharge, \( Q = A \times V \)
- Hydraulic radius, \( R = A / P \)

Steady-uniform flow …

- Normal depth \( \rightarrow \) depth based on uniform flow

From Manning’s formula

\[ Q = \frac{1}{n} \frac{A^{3/2}}{P^{1/2}} S_0^{1/2} \]

\[ \Rightarrow \frac{A^{5/3}}{P^{2/3}} = \frac{nQ}{\sqrt{S_0}} \]

Rectangular channel

\[ \left( \frac{Bh_n}{B + 2h_n} \right)^{5/3} = \frac{nQ}{\sqrt{S_0}} \]

\[ \Rightarrow h_n = \left( \frac{nQ}{B \sqrt{S_0}} \right)^{2/3} \]

Wide rectangular channel

\[ \left( \frac{B h_n}{B + 2h_n} \right)^{2/3} = \frac{nQ}{\sqrt{S_0}} \]

\[ \Rightarrow h_n = \left( \frac{nQ}{B \sqrt{S_0}} \right)^{1/3} \]

Can you work out similarly the relationship for the normal depth using the Chezy’s formula?
Uniform flow computation in a compound channel

- The flow velocity and the carrying capacity of the channel are normally different for the main channel and the over bank flow or the floodplain.
- The roughness coefficient may also vary in the main channel and the floodplain.

![Diagram of a compound channel with labels A1, A2, A3, h1, h2, h3, b11, b12, b13, b21, b22, b31, b32, and total discharge equation.]

The usual approach is

\[ Q_{\text{total}} = Q_1 + Q_2 + Q_3 = \sum \frac{1}{n_j} A_j R_j^{2/3} S^{1/2} \]

That is, the total discharge is the sum of the discharges through each segment of the cross section.

Specific energy

- The energy (in water head) due to the depth of water \((h)\) plus its velocity (the velocity head) is the specific energy.

\[ E = h + \frac{V^2}{2g} = h + \frac{Q^2}{2gA^2} \]

- Differentiating w.r.t. \(h\)

\[ \frac{dE}{dh} = 1 - \frac{Q^2}{gA^3} \frac{dA}{dh} \]

\[ \frac{dE}{dh} = 1 - \frac{V^2}{gD} \]

- With \(dA/dh = B\) and \(D = A/B_{1/3}\)

For a given discharge there exists a min. specific energy \(E_{\text{min}}\) (at \(dE/dh = 0\)):

\[ \frac{V}{\sqrt{gD}} = 1 \quad \Rightarrow \quad \frac{V}{\sqrt{gh}} = 1 \]

For a rectangular channel
Critical depth, Froude number

- Water depth at $E_{\text{min}}$ is called a “critical depth”, $h_c$.

$$\frac{V}{\sqrt{gD}} = 1 \quad \Rightarrow \quad A^2 D = \frac{Q^2}{g}$$

$$h_c = \left( \frac{Q^2}{gB^2} \right)^{\frac{1}{3}} = \left( \frac{q^2}{g} \right)^{\frac{1}{3}}$$

- Froude number (Fr) represents the ratio of inertia forces to gravity forces, given by

$$Fr = \frac{V}{\sqrt{gD}} \quad \Rightarrow \quad \text{At critical depth} \ (h_c), \ Fr = 1.$$

Subcritical and supercritical flows

Subcritical flow: \quad Fr < 1 \quad \Rightarrow \quad V < \sqrt{(gD)}

Critical flow: \quad Fr = 1 \quad \Rightarrow \quad V = \sqrt{(gD)}

Supercritical flow: \quad Fr > 1 \quad \Rightarrow \quad V > \sqrt{(gD)}
Celerity, subcritical and supercritical flow

- The critical velocity $\sqrt{gD}$ is also the speed, or more precisely, “celerity” of small gravity waves that occur in shallow water in open channels as a result of a disturbance to the free surface.

$$c = \sqrt{gD} \quad \Rightarrow \quad c = \sqrt{gh}$$

- A gravity wave can be propagated upstream in water of subcritical flow ($c > V$), but not in water of supercritical flow ($c < V$).

Wave patterns created by disturbances: (a) Still water, $V = 0$; (b) subcritical flow, $c > V$; (c) critical flow, $c = V$; and (d) supercritical flow, $c < V$.

Non-uniform, steady flow

Gradually varied flow- backwater curve computation

(non-uniform, steady flow)
Gradually varied (non-uniform) steady flow

- Gradually varied (non-uniform) steady flow equations can be derived either by

  (1) Taking $\frac{\partial}{\partial t}$ terms equal to zero (because steady flow) in the St. Venant equation, or

  (2) From the consideration of total energy (Bernoulli equation) between two sections in a uniform channel.

---

\[
\frac{dh}{dx} = \frac{S_0 - S_f}{1 - Fr^2} \quad \text{Manning's formula}
\]

\[
Fr^2 = \frac{B_T Q^2}{g A^3}
\]

OR

\[
\frac{dh}{dx} = \frac{S_0 - S_f}{dE/dh}
\]

---

Exercise:

Rewrite the formula for $S_f$ for a wide rectangular channel. Can you workout $S_f$ using Chezy's formula?
Gradually varied (non-uniform) steady flow

Equation in terms of normal and critical depths for a wide-rectangular channel:

\[ \frac{dh}{dx} = S_0 \frac{h^3 - h_n^3}{h^3 - h_c^3} \]

\[ h_n^3 = \frac{Q^2}{C^2 B^2 S_0} \]

\[ h_c^3 = \frac{Q^2}{gB^2} \]

C = Chezy’s coefficient, B = constant bed width, h_n = normal depth, h_c = critical depth.

Gradually varied flow – flow profiles

<table>
<thead>
<tr>
<th>Region</th>
<th>y &gt; y_n</th>
<th>S_f &lt; S_o</th>
<th>Fr^2 &lt; 1</th>
<th>(\frac{dy}{dx}) is positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Region 1</td>
<td>y &gt; y_n</td>
<td>S_f &lt; S_o</td>
<td>Fr^2 &lt; 1</td>
<td>(\frac{dy}{dx}) is positive</td>
</tr>
<tr>
<td>Region 2</td>
<td>y_c &gt; y &gt; y_n</td>
<td>S_f &gt; S_o</td>
<td>Fr^2 &lt; 1</td>
<td>(\frac{dy}{dx}) is negative</td>
</tr>
<tr>
<td>Region 3</td>
<td>y &gt; y_c &gt; y</td>
<td>S_f &gt; S_o</td>
<td>Fr^2 &gt; 1</td>
<td>(\frac{dy}{dx}) is positive</td>
</tr>
</tbody>
</table>
Solution of gradually varied steady flow equation

- Numerical solution (direct step solution)

\[
\frac{dh}{dx} = \frac{S_0 - S_f}{1 - Fr^2}
\]

The problem here is \( S \) and \( Fr \) are dependent on \( h \), i.e. the equation is implicit on \( h \).

\[
\Delta x = \Delta h \frac{1 - Fr^2}{S_0 - S_f}
\]

Starting at the downstream boundary, compute the horizontal distance \( \Delta x \) that corresponds to a given change in depth \( \Delta h \).

OR

\[
\Delta h = \Delta x \frac{S_0 - S_f}{1 - Fr^2}
\]

Starting at the downstream boundary, compute change in depth \( \Delta h \) for a chosen horizontal distance \( \Delta x \).

Solution of gradually varied steady flow equation

- Numerical solution (improved Euler’s method)

\[
\frac{dh}{dx} = \frac{S_0 - S_f}{1 - Fr^2}
\]

Remember!

When starting from downstream towards upstream, make sure that \( \Delta x \) is negative.

\[
h_{j+1/2} = h_j + \frac{\Delta x}{2} \left[ \frac{S_0 - S_f}{1 - Fr^2} \right] \Delta h_j
\]

\( h_j = \) water depth at section \( j \), \( h_{j+1} = \) water depth the section next to section \( j \), \( h_{j+1/2} = \) water depth between sections \( j \) and \( j+1 \).

\[
h_{j+1} = h_j + \Delta x \left[ \frac{S_0 - S_f}{1 - Fr^2} \right] \Delta h_{j+1/2}
\]

\( \triangleleft \) Compute \( S \) and \( Fr \) with \( h = h_j \)

\( \triangleleft \) Compute \( S \) and \( Fr \) with \( h = h_{j+1/2} \)
Solution of gradually varied steady flow equation

- Numerical solution (Predictor-corrector method)

\[
\frac{dh}{dx} = \frac{S_0 - S_f}{(1 - Fr^2)}
\]

\[
\uparrow
\]

Predictor: \( h_{j+1}^{pre} = h_j + \Delta x \left[ \frac{S_0 - S_f}{1 - Fr^2} \right] \)

\[
\downarrow
\]

Corrector: \( h_{j+1}^{cor} = h_j + \Delta x \left[ \frac{S_0 - S_f}{1 - Fr^2} \right] \)

\[
\left( \frac{S_0 - S_f}{1 - Fr^2} \right)_{h_j} + \left( \frac{S_0 - S_f}{1 - Fr^2} \right)_{h_{j+1}^{pre}}
\]

Repeat ‘Corrector’ step with a new value of ‘predictor h’ until the ‘predictor h’ is close enough to ‘corrector h’. The new value of predictor is normally taken as the current value of the corrector h.

Remember!
When starting from downstream towards upstream, make sure that \( \Delta x \) is negative.

Hydraulic jump

- Occurs when supercritical flow changes into a subcritical flow.

\[
\frac{h_2}{h_1} = \frac{1}{2} \left( \sqrt{1 + 8 Fr_1^2} - 1 \right)
\]

- Practical application: to dissipate energy downstream of dams, weirs, etc.

Energy loss, \( \Delta E = E_1 - E_2 = \frac{(h_2 - h_1)^3}{4h_1h_2} \)
Unsteady, non-uniform flow

Unsteady Flow, St. Venant Equation

Flood Wave Propagation

Unsteady non-uniform flow: moving flood wave

Longitudinal view of water surface in an open channel
Unsteady non-uniform flow

Longitudinal Profile: Water surface profile along a river channel at a given time. → Represents variation in space

Hydrograph: Discharge at various times through a given section of a river. → Represents variation in time

Computation of unsteady non-uniform flow considers variation in both space and time together.

Saint Venant equations for 1D channel flow

- Assumptions of Saint-Venant equation
  - One-dimensional flow
  - Gradually varying flow in space and time
  - Hydrostatic pressure prevails, vertical acceleration is negligible
  - Stable and relatively small bed slope
  - Longitudinal axis of the channel is approximated as a straight line.
  - Incompressible, homogeneous and constant viscosity fluid

- Consists of “Continuity” and “Momentum” equations
- Momentum equation can be derived from Euler equation with an added bed shear force.
Saint Venant equations …

- Momentum force:
  \[ \rho \left( \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} \right) A \delta x \]

- Gravity force component:
  \[ \rho g A \delta x S_0 \]

- Pressure force:
  \[ - \rho g \frac{\partial h}{\partial x} \delta x A \]

- Shear force due to friction:
  \[ - \rho g A \delta x S_f \]

Saint Venant equations …

- Equating momentum and applied forces:
  \[ \frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial h}{\partial x} - g \left( S_0 - S_f \right) = 0 \]

\[ V = \frac{Q}{A} \]

\[ \frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( Q \frac{Q^2}{A} \right) + g A \frac{\partial h}{\partial x} - g A \left( S_0 - S_f \right) = 0 \]
Saint Venant equations …

- Various flood wave approximations from the Saint-Venant equation:

Continuity equation
\[
\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0
\]

Momentum equation
\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA\frac{\partial h}{\partial x} - gA(S_0 - S_f) = 0
\]

- Kinematic wave
- Diffusion wave
- Full dynamic wave

Kinematic wave approximation

From continuity equation: \( \frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \)

Kinematic wave approximation neglects acceleration and pressure terms of the momentum equation, i.e.

\( S_0 = S_f \)

This assumption allows to use Manning or Chezy equation:

\[
Q = \frac{1}{n} A R^{2/3} S^{1/2} \quad \text{OR} \quad Q = C A R^{1/2} S^{1/2}
\]

Note: If there is lateral inflow \( q \), the continuity equation becomes:

\[
\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q
\]
Kinematic wave approximation

In both Manning and Chezy equation, A can be expressed in terms of Q as

\[ A = \alpha Q^\beta \]

Using Manning equation, the coefficients alpha and beta are given by

\[ \alpha = \left( \frac{nP^{2/3}}{S^{1/2}} \right)^{0.6} \quad \text{and} \quad \beta = 0.6 \]

**Exercise:**
Verify that the given \( \alpha \) and \( \beta \) are for the Manning equation and derive them for the Chezy equation.

---

Kinematic wave approximation

In continuity equation, replacing \( A \) with \( \alpha Q^\beta \) and simplifying.

\[ \frac{\partial Q}{\partial x} + \alpha \beta Q^{\beta - 1} \frac{\partial Q}{\partial t} = 0 \]

\[ \frac{\partial Q}{\partial x} + \frac{1}{c_k} \frac{\partial Q}{\partial t} = 0 \]

\( c_k \) is the kinematic wave celerity.

Where

\[ c_k = \frac{1}{\alpha \beta Q^{\beta - 1}} = \frac{1}{\beta} \frac{Q}{\alpha Q^\beta} = \frac{5}{3} \frac{Q}{A} = \frac{5}{3} V \]
Unsteady flow and flood wave propagation

A note on numerical methods for the solution
1D unsteady flow

- In practice almost always solved using numerical methods.

- “Finite Difference Method” is one of the widely used numerical methods.

- Discretization is used in space and time to approximate the “partial difference equation” to a “finite difference (algebraic) equation”.
Finite difference numerical solution ...

- Understanding space-time discretisation of numerical schemes

![Diagram showing space-time discretisation]

- Explicit and Implicit schemes:
  - In an explicit scheme the value of the variable at time level $n+1$ can be computed directly (or explicitly) from the values at time level $n$.
  - In an implicit scheme the computation of the value of a variable at $n+1$ involves one or more of the values from the same time level (i.e. $n+1$).
Numerical solution of kinematic wave (an example)

An example of an explicit scheme

\[ \frac{\partial Q}{\partial x} + \frac{1}{c_k} \frac{\partial Q}{\partial t} = 0 \]

\[ \frac{Q_{j+1}^n - Q_j^n}{\Delta x} + \frac{1}{c_k} \frac{Q_{j+1}^{n+1} - Q_{j+1}^n}{\Delta t} = 0 \]

\[ Q_{j+1}^{n+1} = C_r Q_j^n + (1 - C_r) Q_{j+1}^n \]

Where

\[ C_r = c_k \frac{\Delta t}{\Delta x} \]

\( C_r \) is the Courant Number.

An example of an implicit scheme

\[ \frac{\partial Q}{\partial x} + \frac{1}{c_k} \frac{\partial Q}{\partial t} = 0 \]

\[ \frac{Q_{j+1}^{n+1} - Q_j^{n+1}}{\Delta x} + \frac{1}{c_k} \frac{Q_{j+1}^{n+1} - Q_{j+1}^n}{\Delta t} = 0 \]

\[ Q_{j+1}^{n+1} = \frac{1}{1 + C_r} Q_{j-1}^n + \frac{C_r}{1 + C_r} Q_j^n \]

Where

\[ C_r = c_k \frac{\Delta t}{\Delta x} \]

\( C_r \) is the Courant Number.
Finite difference numerical solution ...

- Stability of a scheme
  - In numerical methods, the choice of the space step ($\Delta x$) and time step ($\Delta t$) is important!
  - An explicit scheme becomes unstable when the Courant number ($C_r = c\Delta t/\Delta x$) is greater than 1. Where $c$ is the celerity.
  - An implicit solution is always stable. However, $C_r >> 1$ is not recommended.

Finite difference numerical solution ...

- Initial and boundary conditions:
  - Numerical schemes are based on the assumption that variable values are known at all points on space at time level zero – also called an initial condition.
  - A numerical scheme also requires that values at all time levels at the starting point on space (called an upstream boundary condition) and/or at the end point on space (called the downstream boundary condition).
Commonly used initial and boundary conditions

- Initial condition:
  - Global or local water depth and steady state computation

- Upstream boundary condition:
  - Discharge hydrograph

- Downstream boundary condition:
  - Q-h relationship

End of Presentation