Hydroinformatics
Module 4: Numerical Methods I

Lecture 1 and 2: Models, ODE, Explicit and implicit methods

I.Popescu

Remarks:

1. Schedule
2. Prerequisites
   - Basic mathematics,
   - Fluid dynamics and fluid mechanics,
   - Mathematical representation of fluid flow equations
3. Marking
   - **Assignments (15%)**
   - **Written examination (30%)**

Course

- Main aim
  - introduce numerical techniques which can be used on computers
  - overview of
    - **WHAT** it can be done
    - **HOW** it can be done
- At the end of lectures you will:
    - apply standard methods to simple flow problems and determine the optimal values for time steps and grid size
    - assess simulation results produced by different methods
    - evaluate the relevance of their use
    - identify and analyse the reason for failure of particular methods and, possibly, provide solutions

1-Why Numerical Methods?

Computational Fluid Dynamics

- Computational Fluid Dynamics (CFD) is the science of predicting fluid flow, heat transfer, mass transfer, chemical reactions, and related phenomena by solving the mathematical equations which govern these processes using a numerical process (that is, on a computer).

Solution of Equations

Phenomena in nature -described in terms of properties that prevail at each point of space and time separately

↓

Formulating PDE's governing the phenomena

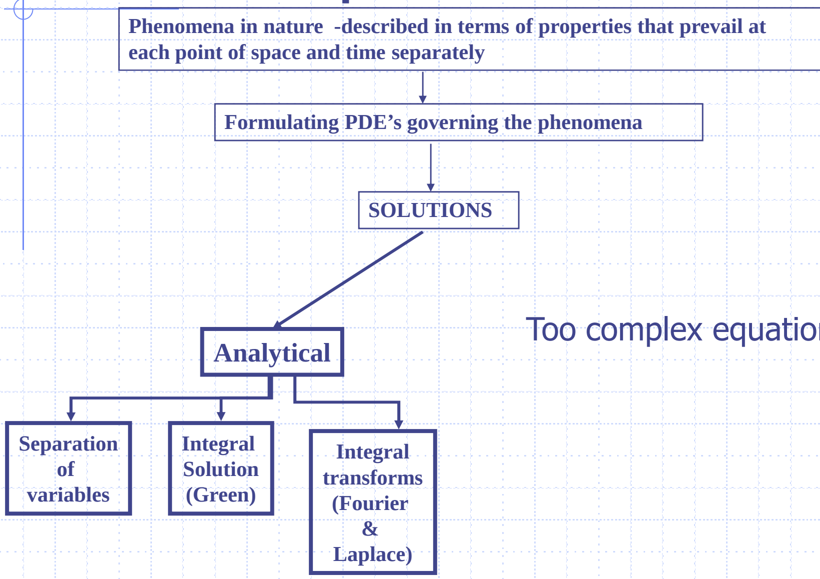
↓

SOLUTIONS

↓

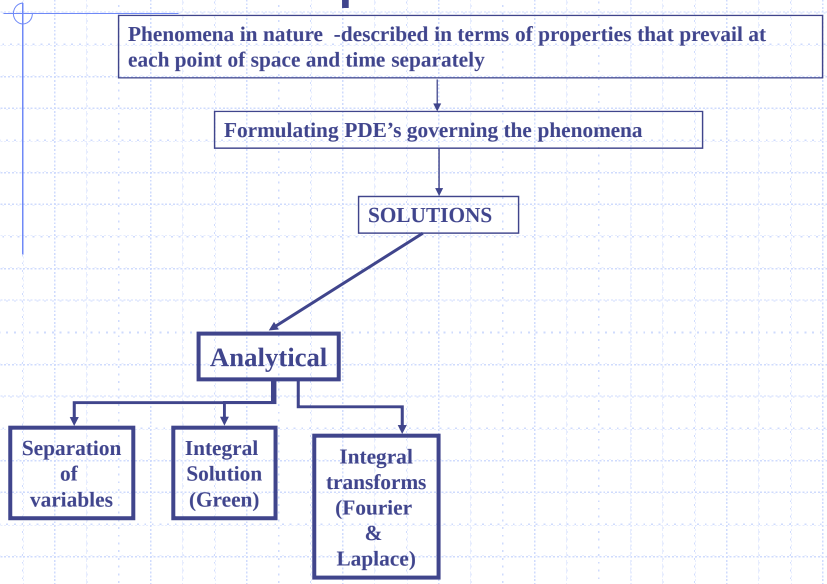
HOW ??

Solution of Equations



Too complex equations?

Solution of Equations



What is a model?

- Is this a model?

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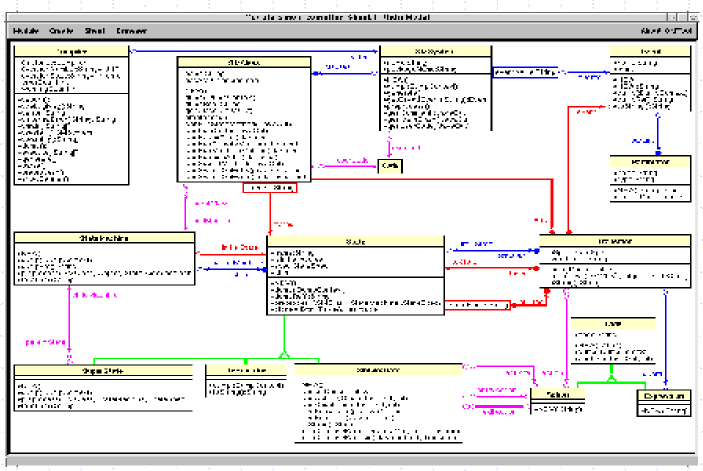
  – Naomi Campbel

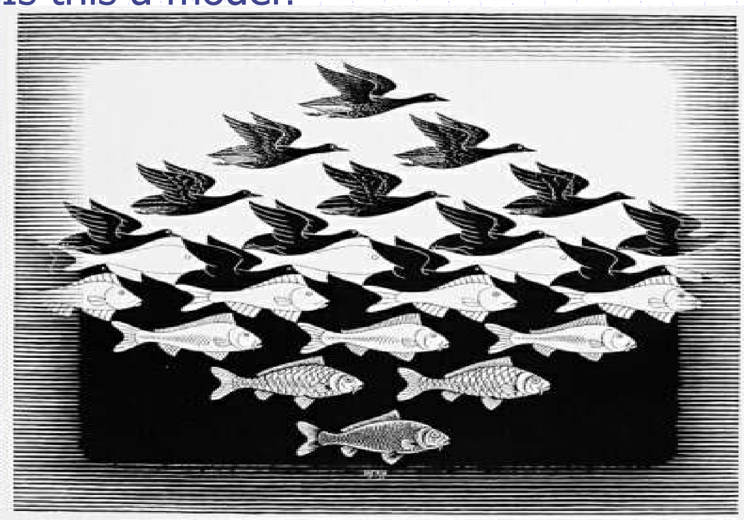
What is a model?

- Is this a model?



What is a model?

- ❑Is this a model?



What is a model?

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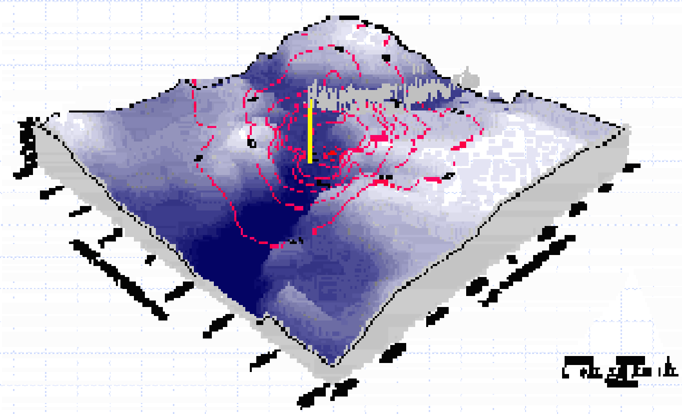
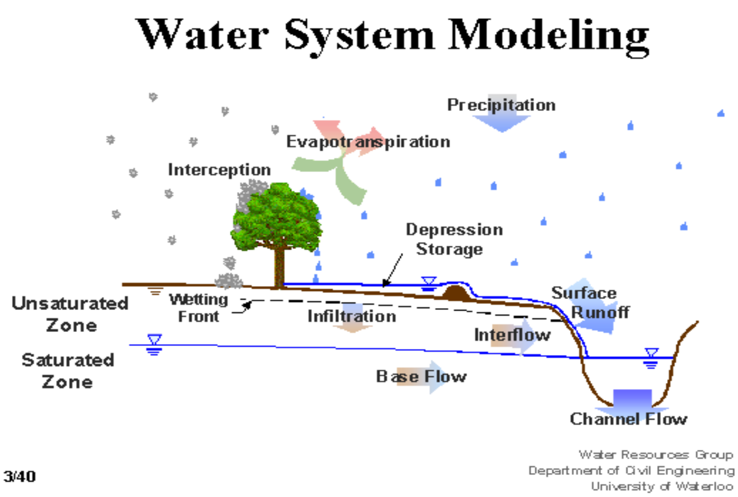


What is a model?



What is a model?

A simplification of reality, but with essential features -- an example:

Reality

Model

What is a model?

- Different forms
- Engineering problem solving
  - A model is a simplified, schematic representations of the real world, a representations that retains enough aspects of the original system to make it useful to the modeller.
- In fluid dynamics
  - physical models (scale models);
  - mathematical models
    - conceptual
    - simulation models

Error and uncertainty in modelling

- Definitions of model:
  - "A system which will convert a given input (geometry, boundary conditions, force, etc.) into an output (flow rates, levels, pressures, etc.) to be used in civil engineering design and operation" (Novak)
  - "A simplified representation of the natural system" (Resfgaard)

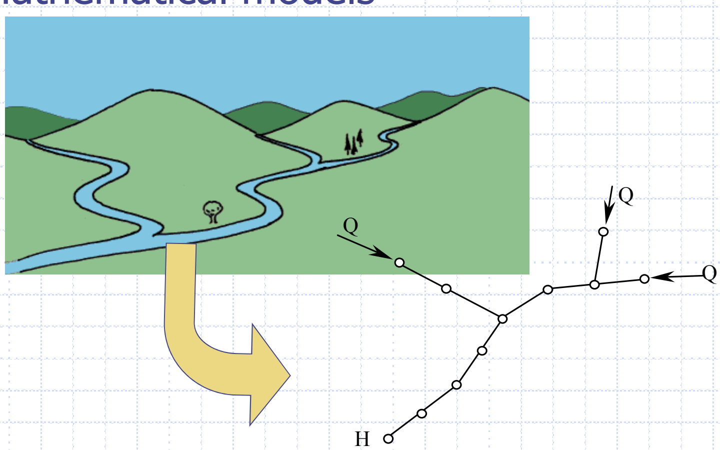
- It is not reasonable to expect that a "simplified representation" convert input into exact output.
- Errors are inevitable.
- The actual values of the output variables are uncertain.

What's the difference between a modeling program, and a model?

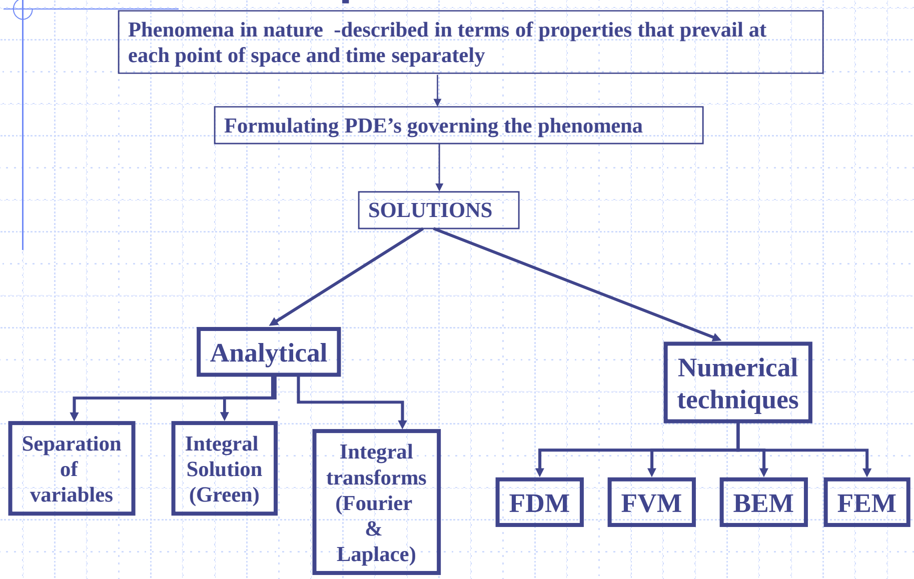
- ❑ Modelling program = software
  - Software with an array of equations that simulate river flow
- ❑ Model = software + data
  - A modelling program + data specific to a certain river

What is a model?

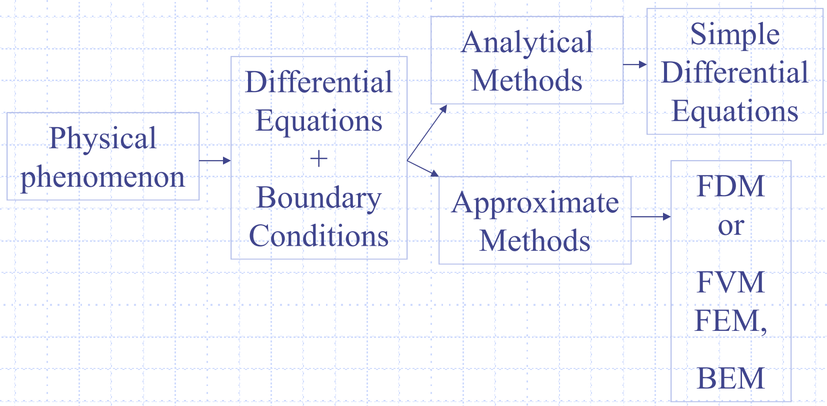
- ❑ Mathematical models



Solution of Equations



Why numerical methods?



- ❑ General differential equations are too complicated to be solved analytically
- ❑ FDM, FEM, FVM, BEM are approximate numerical approaches for solving differential equations

Why Numerical Methods?

- ❑ Analytical solutions
  - derived only for few engineering problems
  - limited number of closed form solutions
  - useful to provide insight to the behaviour of an engineering system
- ❑ Numerical solutions
  - fill the gap between theory and experiment

Why Numerical Methods?

- ❑ Easy input - preprocessor.
- ❑ Solves many types of problems
- ❑ Modular design
  - fluids, dynamics, heat, etc.
- ❑ Can run on PC's now.
- ❑ Relatively low cost, even sometimes cheaper

Why Numerical Methods?

- Models are no longer restricted to closed solution problems
- Represent accurately realistically complex system such as:
  - non-linear systems
  - time dependent state variables
- Sometimes physical experiments are impossible to be simulated

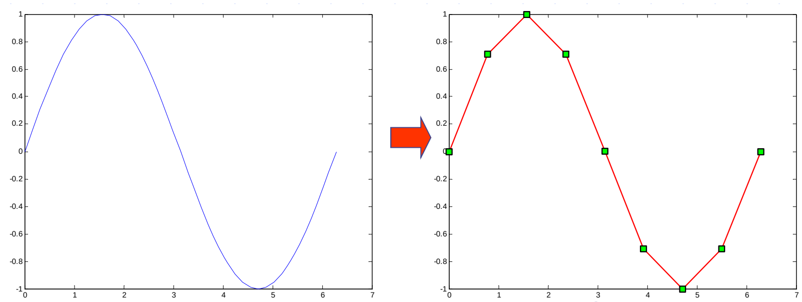
Simple Mathematical Models

$$Dependent\ variable = f\left(independent\ variable,\quad parameters,\quad forcing\ \ function\right)$$

- Dependent variable
  - A characteristic that usually reflects the behavior or state of the system
- Independent variable
  - Are usually dimensions, such as time and space
- Parameters
  - Are reflective of system's properties or compositions
- Forcing functions
  - External influences acting on the system

Basic principle of Numerical Methods

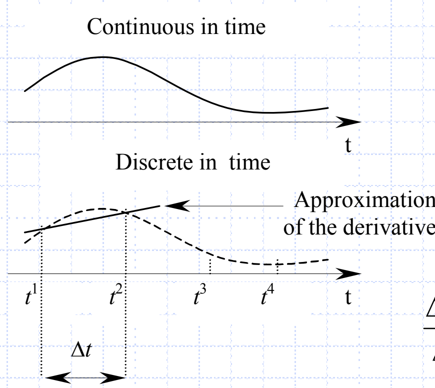
- DISCRETISATION
  - An exact ***'continuous'*** solution is represented by an approximate ***'discrete'*** solution at a number of points.



Exact solution Discretised approximation

Basic principle of Numerical Methods

- DISCRETISATION of space and time

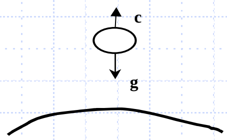


$$\frac{\Delta U}{\Delta t} = \frac{U(t^2) - U(t^1)}{t^2 - t^1} = \frac{U^2 - U^1}{t^2 - t^1}$$

Example - simple mathematical model

- Newton Second Law

$F = ma \quad or \quad a = \frac{F}{m}$



$$v(t^{i+1}) = v(t^i) + \left[g - \frac{c}{m} v(t^i)\right] (t^{i+1} - t^i)$$

new value = old value + slope x step size

Analytical solution - $v(t) = \frac{gm}{c}\left[1 - e^{-ct/m}\right]$

Example-simple mathematical model

- Newton Second Law- Falling object

M=68.1 kg

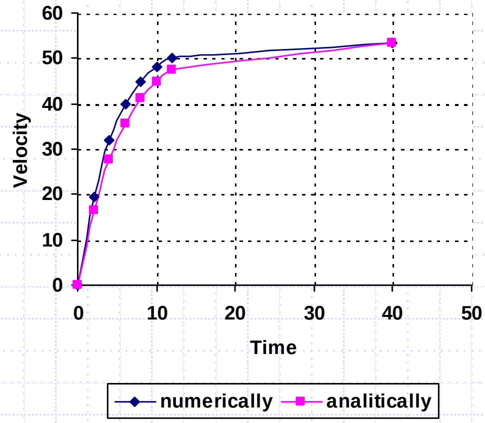
c=12.5 kg/s

t step= 2 sec

Time (s)	Computed velocity (m/s)	
	Numerically	Analytically
0.0	0.000	0.000
2.0	19.600	16.404
4.0	32.005	27.769
6.0	39.856	35.641
8.0	44.824	41.095
10.0	47.969	44.873
12.0	49.959	47.490
:	53.390	53.390

Example - simple mathematical model

❑Newton Second Law- Falling object



M=68.1 kg

c=12.5 kg/s

t step= 2 sec

2- Differential equations

- Quick review -

Introduction and First Definitions[1]

- Differential equations are equations that contain derivatives
  - In an algebraic equation, the unknown is a number. In a differential equation, **the unknown is a function.**
  - Most often the unknown function we seek, expresses the behavior of the state variable or variables as a function of time. This function it is called **state equation**
- The order of the differential equation is the order of the derivative of the unknown function involved in the equation.

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Introduction and First Definitions[2]

- A linear differential equation of order n is a differential written in the following form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

where $a_n(x)$ is not the zero function.

Note: Some books may use the notation y′, y″, y‴, $y^{(4)}$, …for the derivatives.

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Linear Differential Equations

- A linear equation obliges the unknown function *y(x)* to have some restrictions. Accepted operation on *y* are:
  - **Differentiating y**
  - **Multiplying y and its derivatives by a function of the variable x.**
  - **Adding what you obtained in previous step and let it be equal to a function of x.**

Solving differential equations (DE)

- A major field of study
  - Existence: Does a differential equation have a solution?
  - Uniqueness: Does a differential equation have more than one solution? If yes, how can we find a solution which satisfies particular conditions?
- Differential equations require integration to be solved
  - If the values of the unknown function y(x) and its derivatives at some point are known the solution to DE is called an initial value problem (in short IVP).
  - If no initial conditions are given, the description of all solutions to the DE is called the general solution
- Some kinds of D.E.'s can be solved exactly; many not
  - for those that cannot, there are various ways to approximate the solution

Differential Equations

Single Variable

We know:

$$\frac{dy}{dt} = F(y)$$

We want to know:

$$y = f(t)$$

Differential Equations

Several Variable

We know:

$$\frac{dy}{dt} = F_1(y, x, ..., q)$$

$$\frac{dx}{dt} = F_2(y, x, ..., q)$$

$$\vdots$$

$$\frac{dq}{dt} = F_n(y, x, ..., q)$$

We want to know:

$$y = f_1(t)$$

$$x = f_2(t)$$

$$\vdots$$

$$q = f_3(t)$$

Methods for solving DEs

- Solving a differential equation can be done in three major ways:
  - analytically,
  - qualitatively,
  - numerically.

Partial derivatives

- When a function changes with more than one variable at once – for example, both time and space – we use *partial derivatives*
- Notation:

$$\frac{\partial y}{\partial t}$$ = means the rate of change in y with t

$$\frac{\partial y}{\partial x}$$ = means the rate of change in y with x

Total time derivative

- The *total time derivative* is used to describe the change in a variable in relation to a moving (as opposed to stationary) object
- Example:

$$\frac{Dy}{Dt} = \frac{\partial y}{\partial t} + u\frac{\partial y}{\partial x}$$

u is velocity

Time & space derivatives: review

Notation:

$\frac{dy}{dt}$ = ordinary derivative: y changes only with t

$\frac{\partial y}{\partial t}$ = partial derivative: change in y with t at a fixed point

$\frac{Dy}{Dt}$ = total time derivative: change in y as one moves about (e.g., in a fluid)

Classification of DE (Clasess)

- dimension of unknown:
  - ordinary differential equation (**ODE**) – unknown is a function of one variable, e.g. y(t)
  - partial differential equation (**PDE**) – unknown is a function of multiple variables, e.g. u(t,x,y)
- number of equations:
  - single differential equation, e.g. y'=y
  - system of differential equations (coupled), e.g. $y_1'=y_2$, $y_2'=-g$
- order
  - nth order DE has nth derivative, and no higher, e.g. y''=-g
- linear & nonlinear:
  - *linear* differential equation: all terms linear in unknown and its derivatives
  - e.g $x''+ax'+bx+c=0$, $x'=t^2x$ – linear; $x''=1/x$ – nonlinear

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Classification of PDEs

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=d\frac{\partial u}{\partial x}+e\frac{\partial u}{\partial y}+fu+g$$

$$\Delta = b^2-4ac$$

- elliptic PDE - b2- 4ac < 0
  - – smooth solutions
- hyperbolic PDE - b2- 4ac > 0
  - – solutions with discontinuities
- parabolic PDE - b2- 4ac =0

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General features of PDEs

- ❑ Elliptic PDEs
  - describe processes that have already reached steady state
  - time-independent (e.g. steady-state aquifer flow or heat diffusion).
- ❑ Hyperbolic PDEs
  - describe time-dependent, conservative physical processes, such as convection, that are not evolving towards steady state (e.g. free surface flow, pipe flow).
- ❑ Parabolic PDEs
  - describe time-dependent, dissipative physical processes, such as diffusion, that are evolving towards steady state. (e.g. transient aquifer flow).

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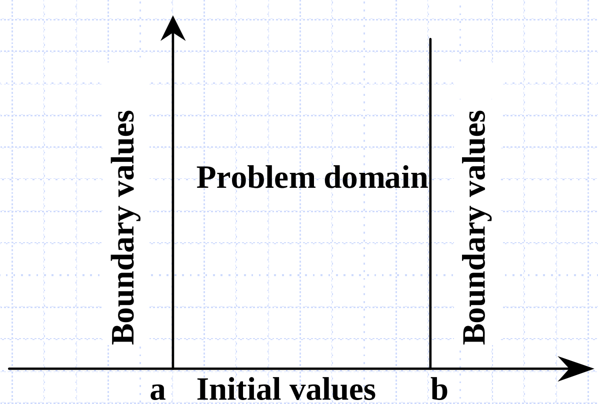
What do we need to solve DE?



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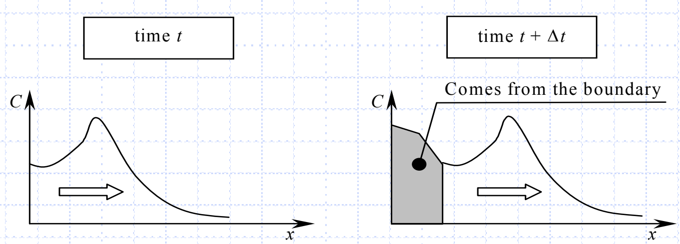
Boundary and initial conditions

- **Initial value problems (IVP)**
  - derivative with respect to time (either dU/dt, or d^2U/dt^2, etc.) in the equation, it is necessary to know the *initial* value of **U** in order to compute its value in the future, within the model domain.
  - initial condition of U must be given.
  - this problems are also called *initial value problems*

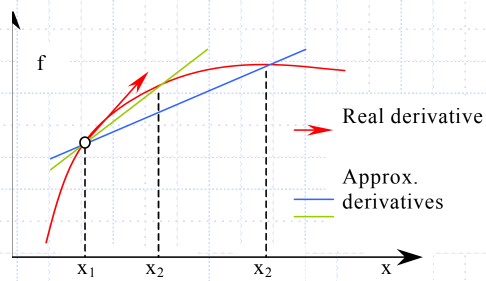
Boundary and initial conditions

- **Boundary value problems (BVP)**
  – Initial conditions are not always sufficient to ensure the existence of a unique solutions. Conditions on the boundary of the domain of the problem must be given as well. These conditions are called *boundary conditions*.



Reminder - Basic formulas

- What is a derivative?
  - Rate of change of a function
  - slope of the tangent line to the graph of a function
  - best linear approximation to a function



$$f'(x) = \lim_{\Delta x \to o} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Δx is small

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Reminder - Basic formulas

- Taylor series expantion

$$f(x \pm \Delta x) = f(x) \pm \Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) \pm \frac{\Delta x^3}{3!} f'''(x) + \frac{\Delta x^4}{4!} f''''(x) \pm \frac{\Delta x^5}{5!} f'''''(x) + .$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{1}{\Delta x}\left[\Delta x f'(x) + \frac{\Delta x^2}{2!} f''(x) + \frac{\Delta x^3}{3!} f'''(x) + .\right] = f'(x) + O(\Delta x)$$

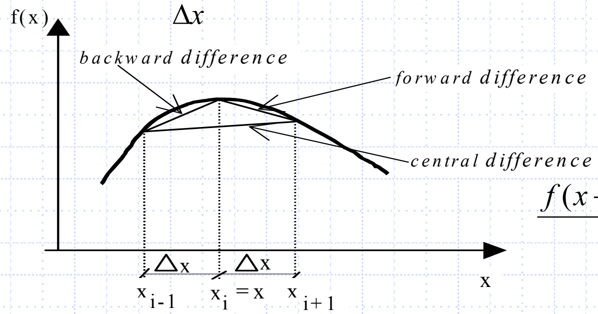
first order accurate in Δx

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = \frac{1}{2\Delta x}\left[2\Delta x f'(x) + 2\frac{\Delta x^3}{3!} f'''(x) + .\right] = f'(x) + O(\Delta x^2)$$

second order accurate in Δx

Reminder - Basic formulas

$$\frac{f(x)-f(x-\Delta x)}{\Delta x}$$



$$\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

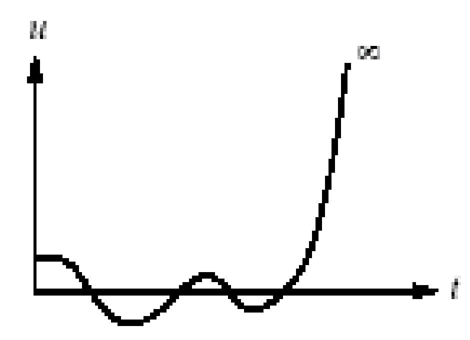
$$\frac{f(x+\Delta x)-f(x-\Delta x)}{2\Delta x}$$

Why study Numerical Methods ?

- ❑NO numerical method is completely trouble free in all situations!
  - ***How should one choose/use an algorithm to get trouble free and accurate answers?***
- ❑No numerical method is error free!,
  - ***What level of error/accuracy do one take in the way the problem is solved?***
- ❑No numerical method is optimal for all types/forms of an equation!
- ❑ ***IMPORTANT :*** In order to solve a physical problem numerically, you must ***understand*** the behavior of the numerical methods used as well as ***the physics of the problem***

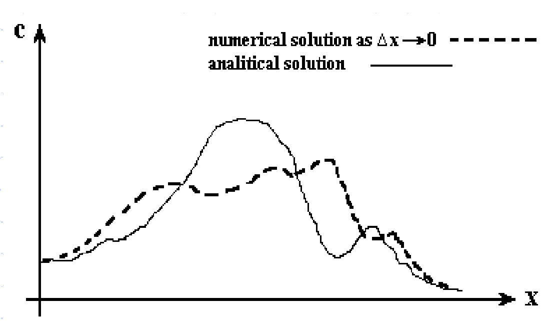
Typical difficulties with NM

- Instability



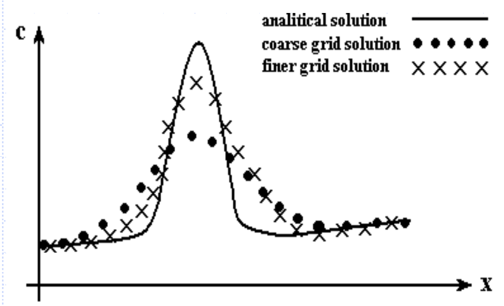
Typical difficulties with NM

- Inconsistency



Typical difficulties with NM

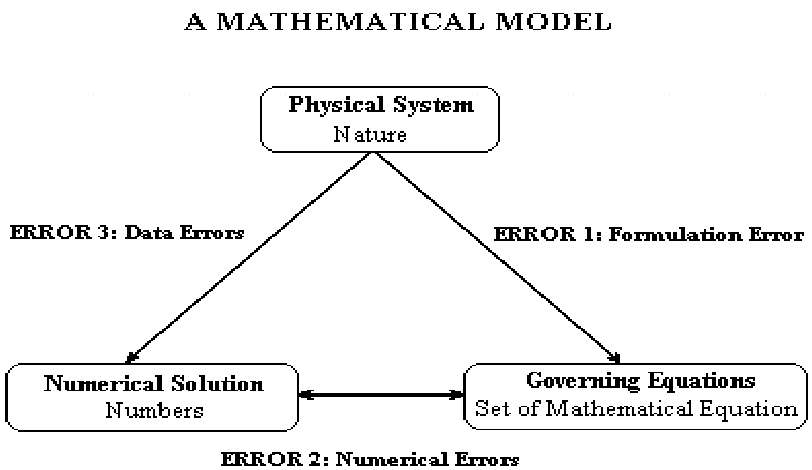
- Inaccurate



Errors

- Errors due to computers:
  - Round-off error
  - Exponent underflow
  - Exponent overflow
- Errors due to numerical methods
  - Truncation error
  - Propagated error
- Measures for errors
  - absolute error
  - relative error

Errors



Hydroinformatician

- ***as a developer and user must understand how a numerical method performs for his/her given problem***
- understanding how various numerical algorithms are derived,
- how the physics effects the numerics
- how the numerics effects the physics
- what are the accuracy/stability properties
- what are the cost of a method for given level of accuracy (this varies substantially from method to method)

Properties of Numerical Methods

❑Convergence

- numerical scheme solution is *convergent* if it comes closer and closer to the analytical solution of the real ODE/PDE when the time step decreases;

❑ **LaxTheorem**:2conditions need it for convergence

- Consistency; and
- Stability

Properties of Numerical Methods

❑Consistency

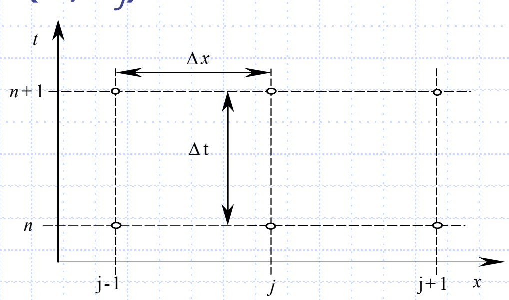
- A scheme is *consistent* if it gives a correct approximation of the ODE/PDE as the time/space step is decreased
- verified using Taylor Series expansion

Properties of Numerical Methods

- Stability
  - A scheme is *stable* if any initially finite perturbation remains bounded as time grows
- Stability verification
  - Matrix method
  - Fourier method
  - Domains of dependence

Notations

- $U^n = U(t^n)$
- $U_j = U(x_j)$
- $U_j^n = U(t^n, x_j)$



4-Finite difference methods for ODE

The general Initial Value Problem[1]

- Given

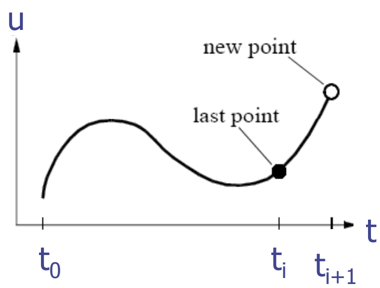
$$\frac{du}{dt} = f(u,t)$$

and initial condition given

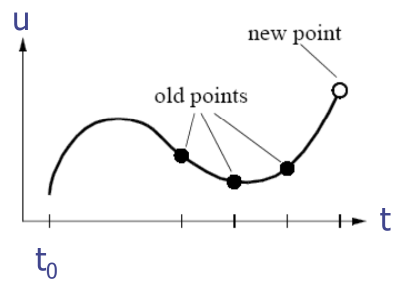
If you know values at time n, find the values at time n+1

The general IVP[2]

- All initial value problems are solved by Integrating forward in t
- There are two main types of integration procedure
  - One-step methods
  - Multi-step methods



One-step method



Multi-step method

General classification

- Numerical scheme
  - a particular discretization of a differential equation
- Schemes:
  - Explicit (Euler method) - the value of the variable at time level n+1 can be computed directly (or explicitly) from the value at time level n.
  - Implicit (Improved Euler) - involves values of the variable at time level n+1, there is an implicit relationship between the derivative and the variable which has to be computed.
  - Mixed (Crank-Nicholson) - schemes in which the explicit and implicit schemes are combined.

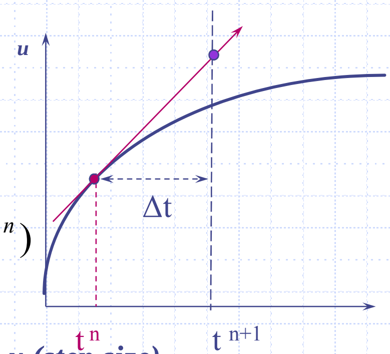
Explicit schemes/Euler/One Step Method

Forward difference approximation

$$\frac{du}{dt} = f(u,t)$$

$$u^{n+1} \approx u^n + \Delta t \cdot f(u^n, t^n)$$

$$u^{final} = u^{initial} + \Delta t \sum_{n=0}^{N-1} f(u^n, t^n)$$

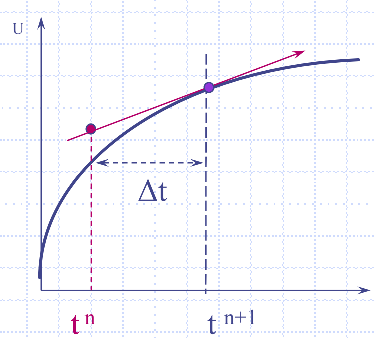


This is the same as saying:
new value = old value + (slope) x (step size)

These schemes are also called ***one step methods*** or "***marching in time",*** because new value is calculated as one step forward from the old value.

Implicit schemes

Backward difference approximation



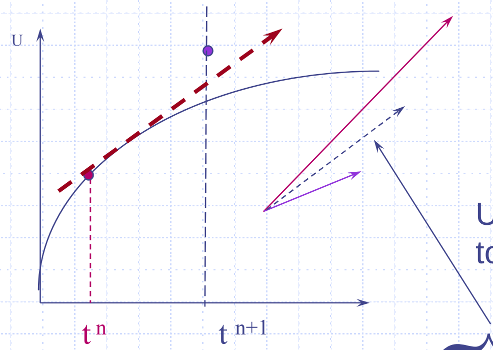
$$\frac{dU}{dt} = f(U,t)$$

$$U^{n+1} - f\left(U^{n+1}, t^{n+1}\right)\Delta t = U^n$$

Iteration need it, ussualy

Mixed schemes

Averaging implicit and explicit schemes

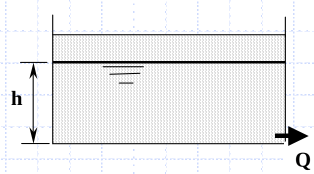


Use this "average" slope to predict U^{n+1}

$$U^{n+1} = U^n + \frac{f\left(U^{n+1}, t^{n+1}\right) + f\left(U^n, t^n\right)}{2} \Delta t$$

Numerical schemes for ODE - Example

Ground water reservoir



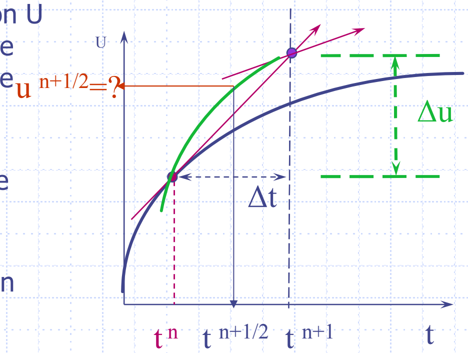
$$\frac{dh}{dt} = -\alpha h$$

-exact solution

$$h = h_o e^{-\alpha t}$$

Multi step methods: Midpoint Methods

- A better estimate of the function U would come from evaluating the derivative at the midpoint of the Δt interval.
- The problem: we know t at the midpoint but we don't know the *u(t)* at the midpoint (yet).
- The solution is to use Euler's method to estimate Δu and then re-estimate Δu using the derivatives evaluated halfway along the line segment encompassing the original Δu.
- This method is called the second order Runge Kutta method, or the midpoint method.



Mixed schemes - Midpoint Methods

$$u^{n+1} = u^n + (a_1 k_1 + a_2 k_2)\Delta t$$

Where for two point schemes

$$k_1 = f(u^n, t^n)$$

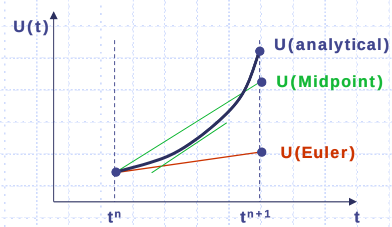
$$k_2 = f(u^n + q_{11} k_1 \Delta t, t^n + p_1 \Delta t)$$

$$a_1, a_2 - weighted \quad coefficients$$

$$a_1 = 1 - a_2;$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$



Exercise

- Apply the Euler and Mid-point methods to the following problem (with known solutions!):

$$\frac{du}{dt} = f(u,t)$$

$$for$$

$$1.\ f(u,t) = t$$

$$2.\ f(u,t) = t^2$$

$$3.\ f(u,t) = u$$

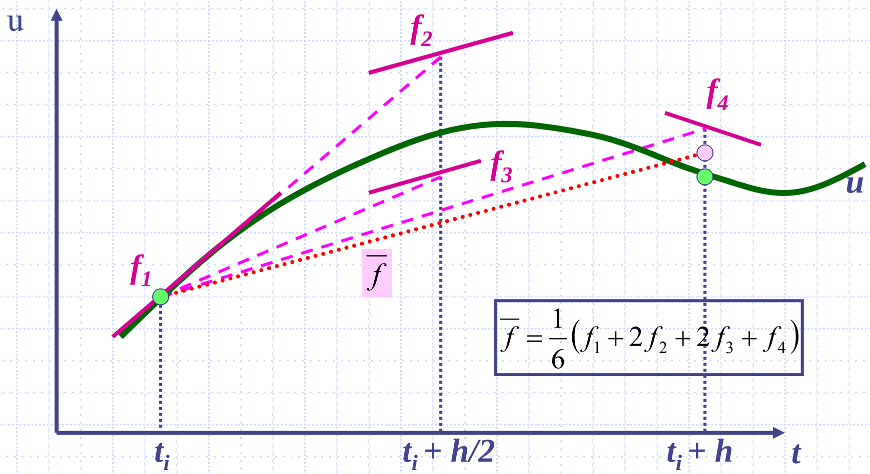
$$4.\ f(u,t) = -u$$

- In each case, compare the errors in the two schemes after taking the same number (10, say) of identical timeteps (start with). Is the mid-point method noticeably superior to Euler?

Mixed schemes - Multi Step Methods

- The midpoint method can be extended by considering other intermediate estimates. The most frequently used variation is the *fourth-order Runge-Kutta* method which considers one estimate at the initial point, two estimates at the midpoint, and one estimate at a trial endpoint.
- This is popular for two reasons
  - It is easy to program
  - It is stable
- However
  - It requires more computing time

Runge-Kutta Method - *4th-order*



$$\bar{f} = \frac{1}{6}\left(f_1 + 2f_2 + 2f_3 + f_4\right)$$

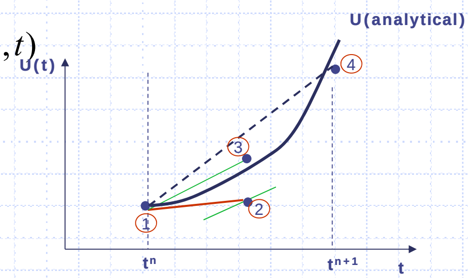
Rungge Kutta - 4 point scheme

$$u^{n+1} = u^n + (a_1k_1 + a_2k_2 + a_3k_3 + a_4k_4)\Delta t$$

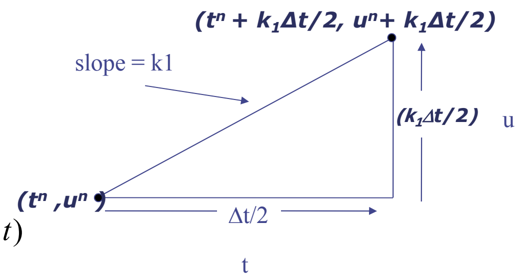
Where

$f(u,t) - the\ derivative\quad at\ (u,t)$

$$k_1 = f(u,t)$$
$$k_2 = f(u + k_1\Delta t/2, t + \Delta t/2)$$
$$k_3 = f(u + k_2\Delta t/2, t + \Delta t/2)$$
$$k_4 = f(u + k_3\Delta t, t + \Delta t)$$



Runge-Kutta 4th order - solution



$f(u,t) - \text{the derivative at } (u,t)$

$$k_1 = f(u,t)$$
$$k_2 = f(u + k_1\Delta t/2, t + \Delta t/2)$$
$$k_3 = f(u + k_2\Delta t/2, t + \Delta t/2)$$
$$k_4 = f(u + k_3\Delta t, t + \Delta t)$$

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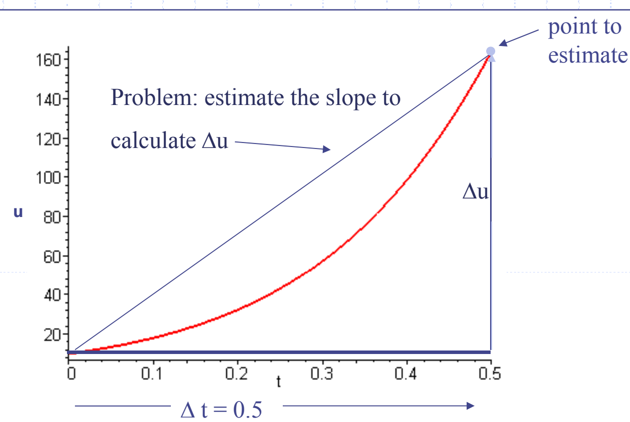
Runge-Kutta versus Euler-example

Solve the equation below using both Euler and Runge Kutta methods

$$\frac{du}{dt} = 6u - 0.007\ u^2$$



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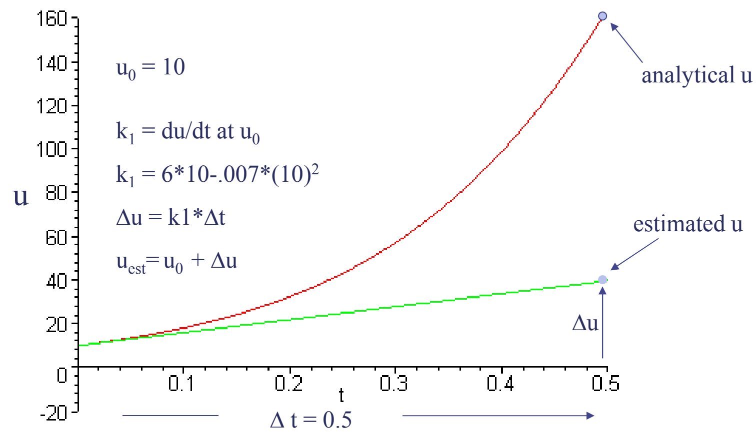
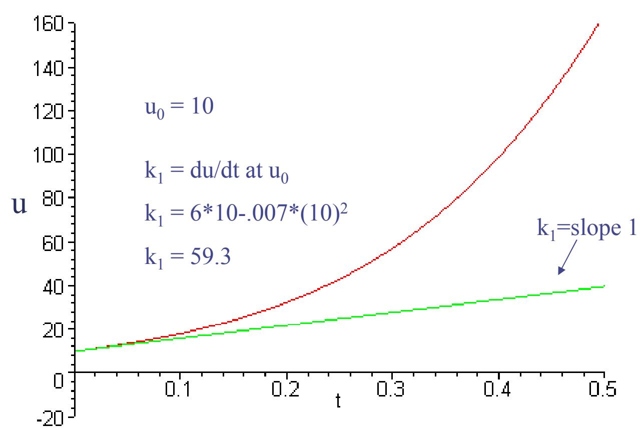
Euler solution

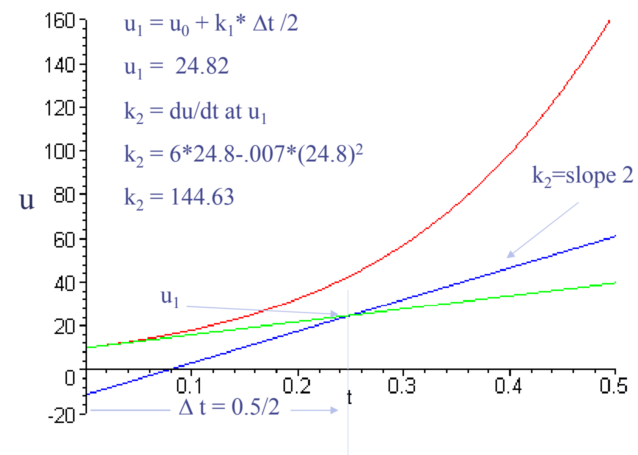


Step 1: Evaluate slope at current value of state variable.



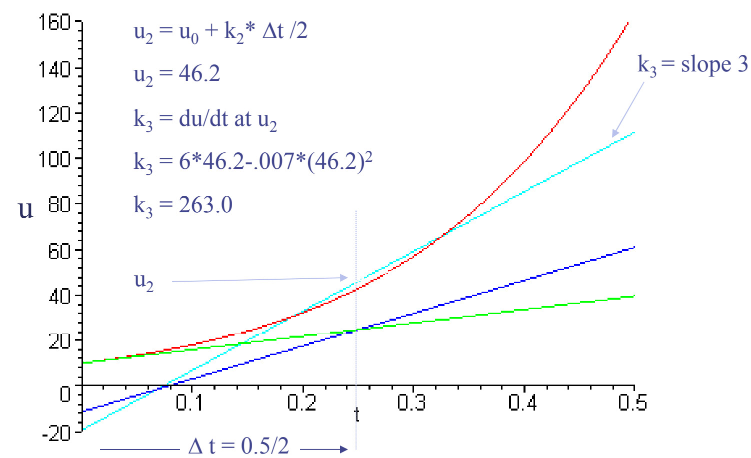
Step 2: Calculate u_1 at $t + \Delta t/2$ using k_1.

Evaluate slope at u_1.



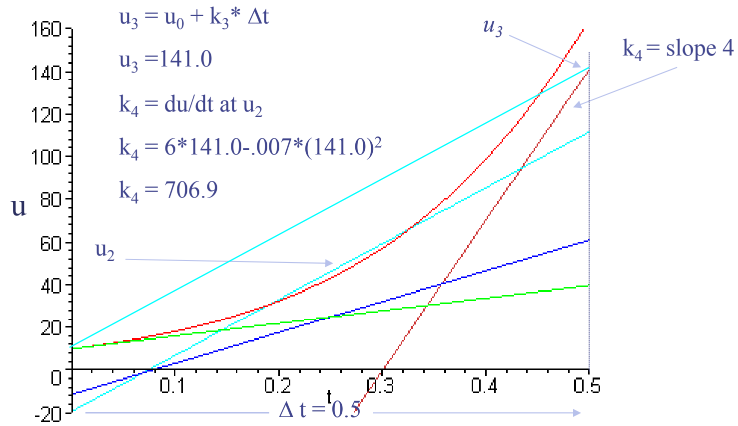
Step 3: Calculate u_2 at $t + \Delta t/2$ using k_2.

Evaluate slope at u_2.



Step 4: Calculate u_3 at $t + \Delta t$ using k_3.

Evaluate slope at u_3.

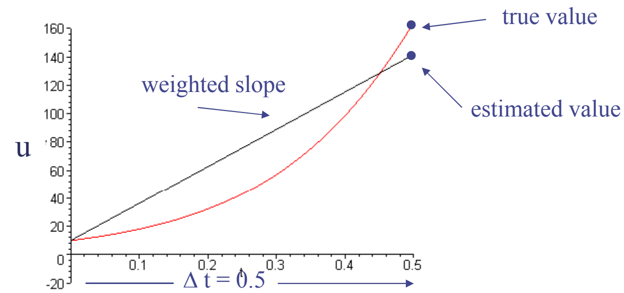


Step 5: Calculate weighted slope.

Use weighted slope to estimate u at $t + \Delta t$

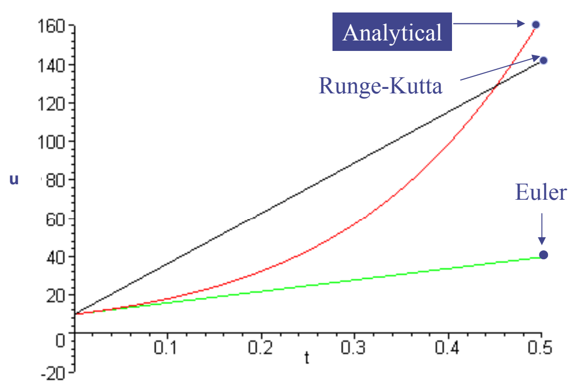
$$\text{weighted slope} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$u^{n+1} = u^n + \Delta t \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$



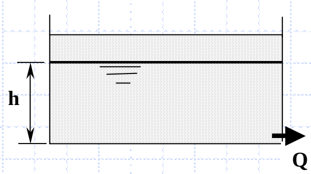
Conclusions

- ***4th order Runge-Kutta offers substantial improvement over Eulers.***
- ***Both techniques provide estimates, not "true" values.***
- ***The accuracy of the estimate depends on the size of the step used in the algorithm.***



Numerical schemes for ODE - Example

Ground water reservoir



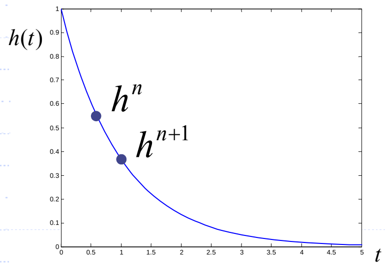
$$\frac{dh}{dt} = -\alpha h$$

-exact solution

$$h = h_o e^{-\alpha t}$$

Numerical schemes for ODE - Example
Solution methods :1. Explicit(Euler)

The right hand side is determined at current time (h^n)



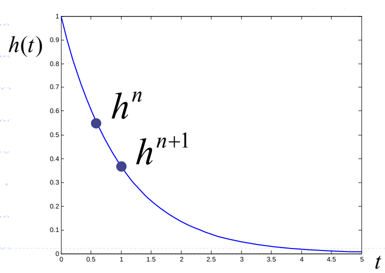
$$\frac{h^{n+1}-h^n}{\Delta t}=-\alpha h^n$$

$$h^{n+1}=h^n(1-\alpha\Delta t)$$

$$\frac{h^{n+1}}{h^n}=(1-\alpha\Delta t)$$

Numerical schemes for ODE - Example
Solution methods :2. Implicit

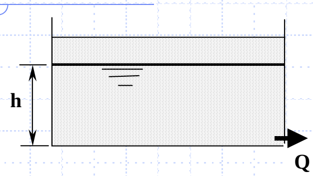
The right hand side is determined at future time (h^{n+1})



$$\frac{h^{n+1}-h^n}{\Delta t}=-\alpha h^{n+1}$$

$$\frac{u^{n+1}}{u^n}=\frac{1}{(1+\alpha\Delta t)}$$

Numerical schemes for ODE - Example



$$\frac{dh}{dt} = -\alpha h$$

Analytic solution: $h = h_o e^{-\alpha t}$

Euler scheme: $h^{n+1} = (1-\alpha\Delta t)h^n$

Implicit scheme: $h^{n+1} = \dfrac{h^n}{1+\alpha\Delta t}$

The BIG question

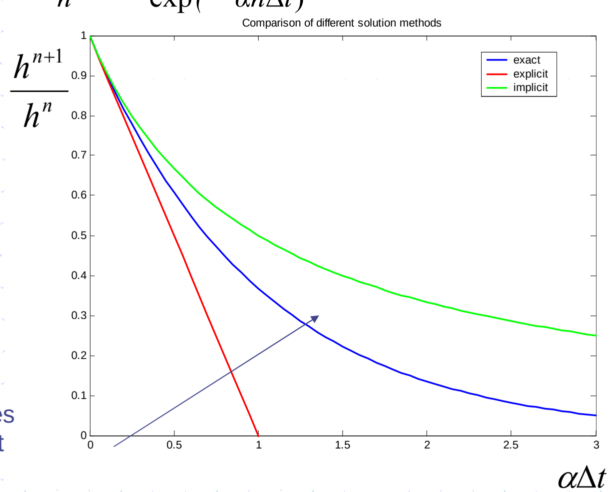
$$\approx \neq =$$

The **BIG** question

- If equations are approximation then it should:
  - behave in the same way as the function to which it is an approximation, and
  - become a better approximation as the time-step is reduced.

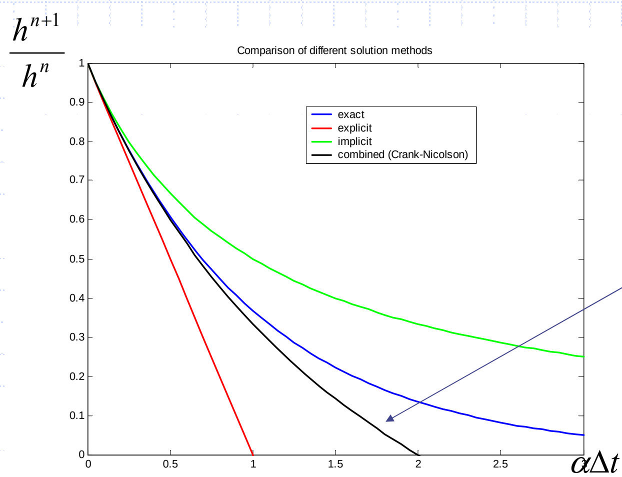
Explicit vs Implicit

Exact solution $$\frac{h^{n+1}}{h^{n}} = \frac{\exp(-\alpha(n+1)\Delta t)}{\exp(-\alpha n\Delta t)} = \exp(-\alpha\Delta t)$$



Exact solution lies between explicit and implicit solutions

Three different solution methods -comparison



Crank-Nicolson is closest to exact solution

Stability, Consistency, Convergence

Lax Theorem:

Consistency + stability ⇔ convergence

This is why you can trust your models !

Consistency : The scheme approximates the ODE more correctly as the discretisation is refined.

Stability: Any initial perturbation remains bounded.

Convergence: The numerical solution becomes closer to the real one as the discretisation is refined

The BIG question

- If equations are approximation then it should:
  - behave in the same way as the function to which it is an approximation, and
  - become a better approximation as the time-step is reduced.

Stability, Consistency, Convergence

Convergence - Explicit method

$$h^n \not\to h^0 e^{-\alpha t}$$

$$h^1 = h^0(1-\alpha\Delta t)$$
$$h^2 = h^1(1-\alpha\Delta t) = h^0(1-\alpha\Delta t)^2$$
$$\ldots$$
$$h^n = h^o(1-\alpha\Delta t)^n$$

$$h^n = \left[1+\left(-\frac{\alpha t}{n}\right)\right]^n h^o$$

Stability, Consistency, Convergence

Convergence - Explicit method (cont)

$$h^n = h^0\left[1^n + 1^{n-1}\left(-\frac{\alpha \cdot t}{n}\right)\binom{n}{1} + 1^{n-2}\left(-\frac{\alpha \cdot t}{n}\right)^2\binom{n}{1} + \ldots\right]$$

where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

$$\frac{n!}{(n-r)!r!} = n(n-1)\ldots(n-r) \to n^r$$

$$\binom{n}{r} = \frac{n^r}{r!}$$

Stability, Consistency, Convergence

Convergence - Explicit method (cont)

Back to the equation

$$h^n \to h^0\left[1^n + \left(-\frac{\alpha \cdot t}{n}\right)\frac{n}{1!} + \left(-\frac{\alpha \cdot t}{n}\right)^2\frac{n^2}{2!} + \ldots\right]$$

$$\to h^0\left[1^n + \left(-\frac{\alpha \cdot t}{1}\right) + \left(-\frac{\alpha \cdot t}{1}\right)^2\frac{1}{2!} + \ldots\right] \to h^0 e^{-\alpha t}$$

Taylor series for $e^{-\alpha \cdot t}$

Stability, Consistency, Convergence

Stability condition

$$\left|\frac{h^{n+1}}{h^n}\right| \le 1 \Longrightarrow -1 \le \frac{h^{n+1}}{h^n} \le 1$$

Stability - Explicit schemes

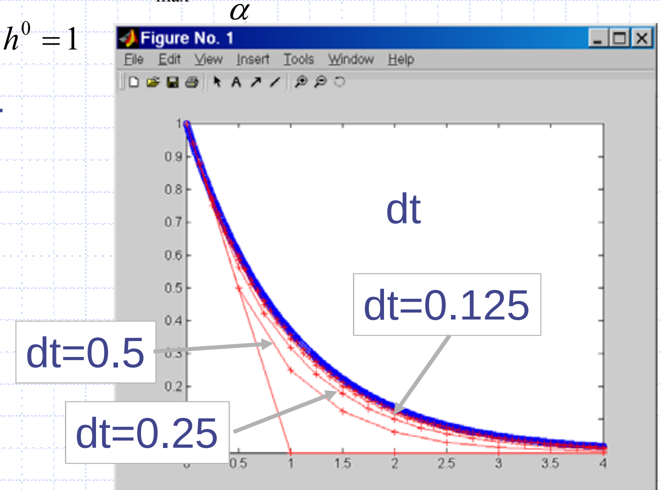
$$-1 \le 1-\alpha\Delta t \le 1 \Rightarrow \begin{cases} -\alpha\Delta t \le 0 \\ \alpha\Delta t \le 2 \end{cases} \Rightarrow \Delta t_{\max} = \frac{2}{\alpha}$$

Stability, Consistency, Convergence

Example $Take \quad \alpha = 1 \Rightarrow \Delta t_{\max} = \frac{2}{\alpha} = 2$

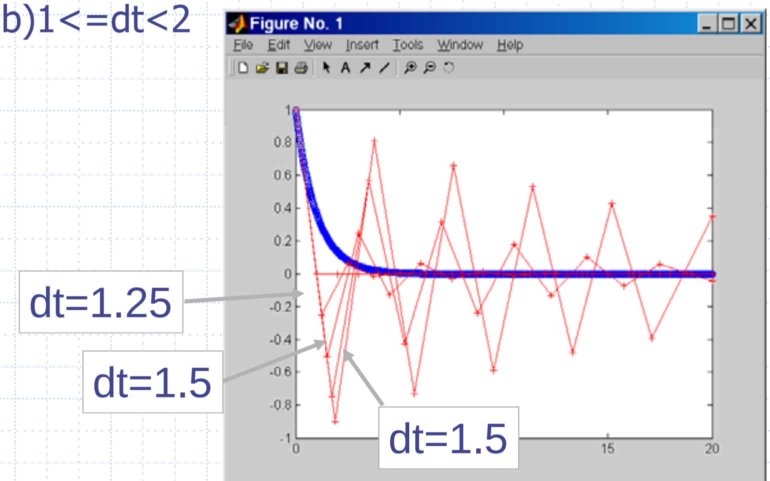
$and \quad h^0 = 1$

a) 0 < dt < 1



Stability, Consistency, Convergence

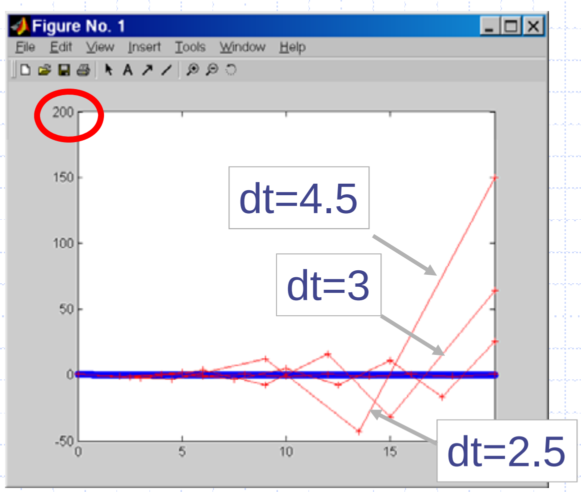
b)1<=dt<2



=>Stable But Oscillatory Approximations

Stability, Consistency, Convergence

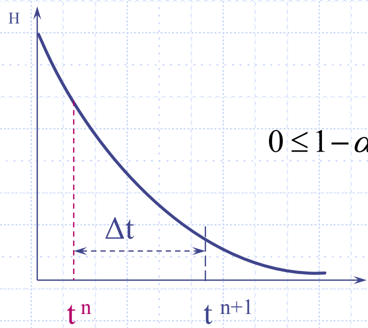
c)2<dt



=>Unstable (i.e. Bad) Approximations

Stability, Consistency, Convergence

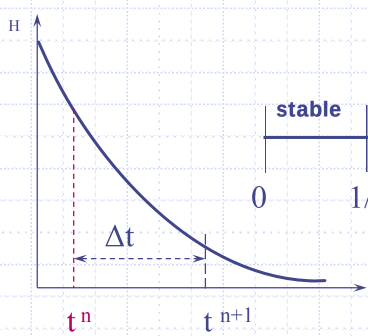
Stability - Explicit schemes

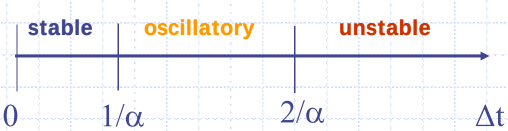


$$0 \le 1 - \alpha \Delta t \le 1 \quad \text{or} \quad 0 \le \Delta t \le \frac{1}{\alpha}$$

Stability, Consistency, Convergence

Stability - Explicit schemes





Stability, **Consistency**, Convergence

Consistency - explicit schemes

$$\frac{dh}{dt}+\alpha h=\underbrace{-\frac{\Delta t}{2}\frac{d^2h}{dt^2}-O\left(\Delta t^2\right)}$$

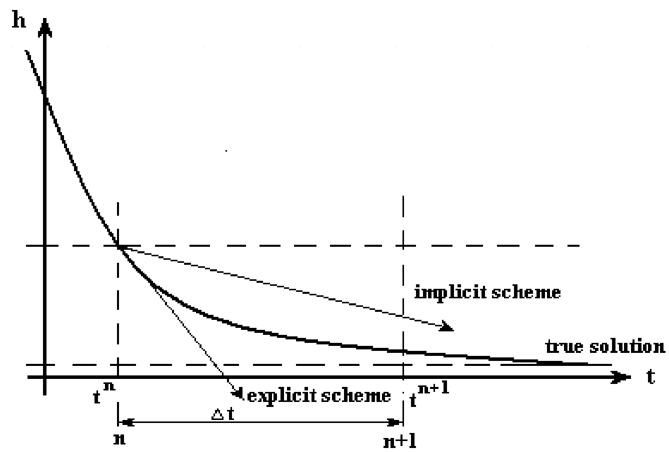
Truncation error - first order accurate

Stability, Consistency, Convergence

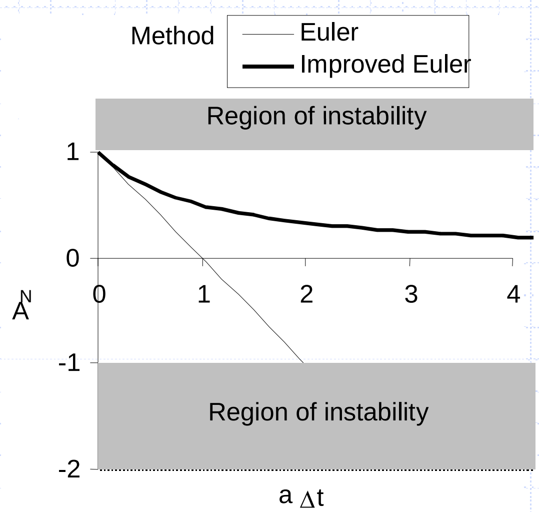
- **Implicit schemes**
  - convergence - FOR YOU TO TEST
  - stability - unconditionally stable
- **Mixed schemes**
  - unconditionally stable

Amplification factor

- U^{n+1}/U^{n} is called the *numerical amplification factor*



Amplitude portrait



4-Finite difference methods for ODE

Quick summary

What you should remember

- ODEs can be solved numerically by replacing the d/dt operator by a difference of the considered function at two time levels.
- When the derivative is estimated using the values of the function at time step level n, the scheme is said to be explicit. When the derivative is estimated using the values at time n+1, the scheme is said to be implicit.
- The discretized ODE is consistent to the real one if it comes closer to the real ODE as dt (or dx) comes closer to zero.

What you should remember

- The difference between the discretised ODE and the real one is called the truncation error. It is obtained by substituting Taylor series expansions into the discretisation.
- The solution is said to be stable if initial perturbations in the numerical solution remain bounded.
- The stability of the solution is investigated by calculating the value of the amplification factor. If $|A^N|$ is smaller than unity, the solution is stable.

What you should remember

- In general, explicit schemes are not stable for all values of Δt (or Δx).
- Implicit schemes are always stable.
- The numerical solution is said to converge to the real one if it comes closer to it as Δt (or Δx) comes closer to 0.
- Consistency and stability of a scheme are necessary and sufficient conditions to the convergence of the solution