Hydroinformatics
Module 4: Numerical Methods I

Lecture 3 and 4: PDE, Hyperbolic PDE, Stability, Accuracy

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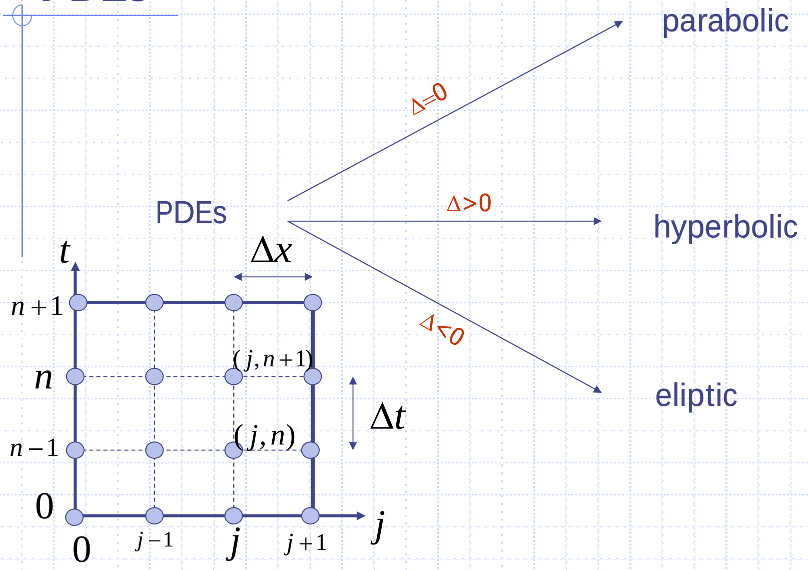
5-Partial Differential Equations

PDEs



Solving PDEs

- ❑Method of characteristics
- ❑Finite Difference method
  - Explicit schemes
  - Implicit schemes
  - Upwind, forward/backward space/time
    - FTCS
    - CTCS
  - Other schemes
    - MoC (1-st and 2-nd order)
    - Preissmann scheme
    - Abbott-Ionescu scheme

PDEs - Examples

- The 1D advection-diffusion equation:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = d\frac{\partial^2 u}{\partial x^2}$$

  - This PDE is first order in time and second order in space.
- Simplify further (drop the second order diffusion or dissipation term):

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

  - This PDE is first order in time and first order in space
  - It describes the time dependent shifting of the function u(x) along x with a velocity a
- Volunteer to solve this equation analytically?

Simple wave equation/ Advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

- Represents
  - Kinematic wave
    - u – water depth (m)
    - a –(celerity (m/s)
  - Transport of a polutant in a river
    - u – concentration (mg/l)
    - a –flow velocity (m/s)
  - Saint Venant equation – a system of two advection equations

Necessary Information to Solve the IBVP

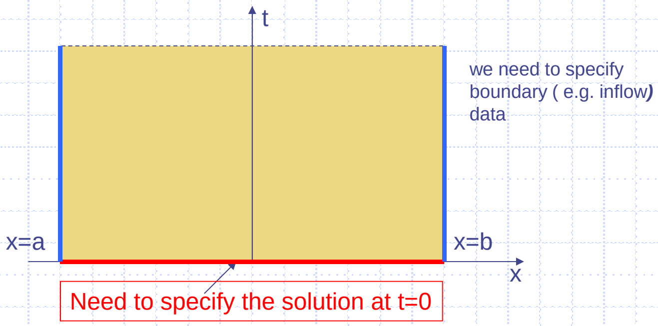
- The **I**nitial, **B**oundary **V**alue **P**roblem represented by the PDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

  - requires extra information in order to to be solvable.
- What do we need?

IBVP

- Because of the hyperbolic nature of the PDE (solution travels from right to left with increasing time), we need to supply:
  - Extent of solution domain
  - What is the solution at start of the solution process: u(x,0)
  - Boundary data: u(b,t), or u(a,t)
  - Final integration time.

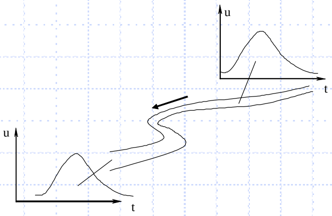


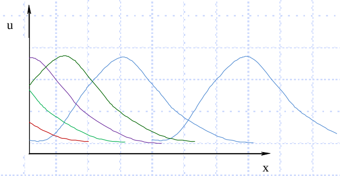
Analytical solution of Hyperbolic PDEs [1]

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 \\ IC: \quad u(x,0) = f(x) \end{cases}$$

Advection equation:

- it describes the time dependent shifting of the function u(x,0) along x with a velocity *a*



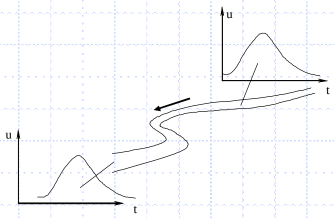


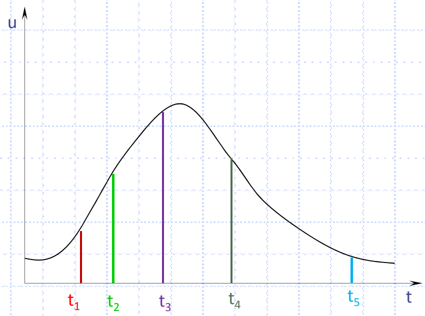
The solution at any time $t > t_0$ can be described as a function of the state at time t_0:

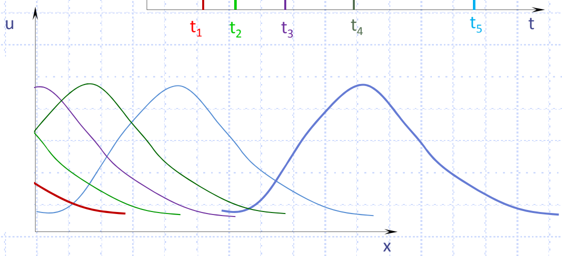
$$u(x + at, t) = u(x,0)$$

This is a so-called *initial value problem in which the state at any time $t > t_0$ can be uniquely* found when the state at time $t = t_0$ is fully given.

Analytical solution of Hyperbolic PDEs [2]



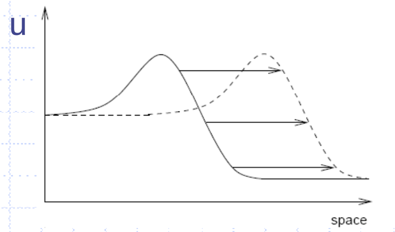




Analytical solution of Hyperbolic PDEs 3

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 \\ IC: \quad u(x,0) = f(x) \end{cases}$$

Advection equation:



$$u(x,t) = u(x - at, 0)$$

1. This is a so-called *initial value problem in which the state at any time $t > t_0$ can be uniquely* found when the state at time $t = t_0$ is fully given.

2. The initial value problem is quite trivial, yet, as we will see below, this problem stands at the basis of numerical methods of hydrodynamics and is numerically surprisingly challenging to solve!

Brief Summary

- There is a checklist of conditions we will need to consider to obtain a unique solution of a PDE:
  1) The PDE
  2) Boundary values (also known as boundary conditions)
  3) Initial values (if there is a time-like variable)
  4) Solution domain

(A)Hyperbolic PDEs

(A)Hyperbolic PDEs

Method of Characteristics

The method of characteristics

- The method of characteristics is a method which can be used to solve the initial value problem (IVP) for general first order PDEs by transforming them in ODEs.
- Consider the first order linear equation :

$$a(x,t)\frac{\partial u}{\partial x}+b(x,t)\frac{\partial u}{\partial t}+c(x,t)u(x,t)=0 \qquad (5.1)$$

- With initial condition $u(x,0)=f(x)$

Goal

- Change coordinates from (x,t) to new coordinate system (x_o,s) so that PDE becomes ODE along certain curves in (x,t) plane.
- Such curves, along which the solution of the PDE reduces to an ODE, are called the ***characteristic curves*** or just the ***characteristics***.
- The new variable s will vary, and the new variable x_0 will be constant along the characteristics. The variable x_0 will change along the initial curve in the x-t plane (along the line t=0).
- How do we find the characteristic curves?

Finding characteristic curves

- Notice that if we choose

$$\frac{dx}{ds} = a(x,t) \quad \text{and} \quad \frac{dt}{ds} = b(x,t) \qquad (5.2)$$

- Then

$$\frac{du}{ds} = \frac{dx}{ds}\frac{\partial u}{\partial x} + \frac{dt}{ds}\frac{\partial u}{\partial t}$$

- And along the characteristic curves

$$\frac{du}{ds} + c(x,t)u(s) = 0 \qquad (5.3)\ \text{-ODE}$$

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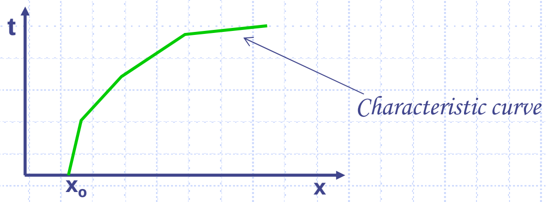
Application steps

- Solve the two characteristic equations, (5.2).
- Find the constants of integration by setting $x(0)=x_0$ (these will be points along the t=0 axis in the x-t plane) and t(0)=0. We now have the transformation from (x,t) to (x_0,s), $x=x(x_0,s)$ and $t=t(x_0,s)$



- Solve the ODE (5.3) with initial condition $u(0)=f(x_0)$, where x_0 are the initial points on the characteristic curves along the t=0 axis in the x-t plane.
- We now have a solution $u(x_0,s)$. Solve for s and x_0 in terms of x and t (using the results of step 1) and substitute these values in $u(x_0,s)$ to get the solution to the original PDE as u(x,t)

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The MoC summary

$$\begin{cases} a(x,t)\dfrac{\partial u}{\partial x}+b(x,t)\dfrac{\partial u}{\partial t}+c(x,t)u(x,t)=0 \\ u(x,0)=f(x) \end{cases}$$

For the selection :

$$\frac{dx}{ds}=a(x,t) \quad \text{and} \quad \frac{dt}{ds}=b(x,t) \quad \text{and} \quad \frac{du}{ds}=\frac{dx}{ds}\frac{\partial u}{\partial x}+\frac{dt}{ds}\frac{\partial u}{\partial t}$$

$\frac{du}{ds}+c(x,t)u(s)=0$ **holds along the characteristic curves** $\frac{dx}{dt}=\frac{a(x,t)}{b(x,t)}$

Cases

- Depending on values of a(x,t), b(x,t) and c(x,t) there are 3 different PDEs:
  - the constant coefficient advection equation,
  - the variable coefficient advection equation,
  - inviscid Burgers' equation.
- For all three examples, the initial conditions are specified as

$$u(x,0)=f(x)$$

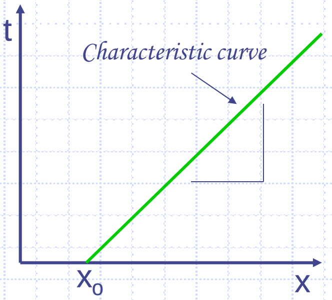
Case 1: c(x,t)=0 and b(x,t)=1

$$\begin{cases} \dfrac{du}{ds} = 0 \\ \dfrac{dx}{dt} = \dfrac{a(x,t)}{b(x,t)} = a(x,t) \\ u(x,0) = f(x) \end{cases} \Rightarrow \begin{cases} u(x,t) = const \\ \text{along} \\ \dfrac{dx}{dt} = a(x,t) \\ u(x,0) = f(x) \end{cases}$$

- **Constant coefficient advection equation**

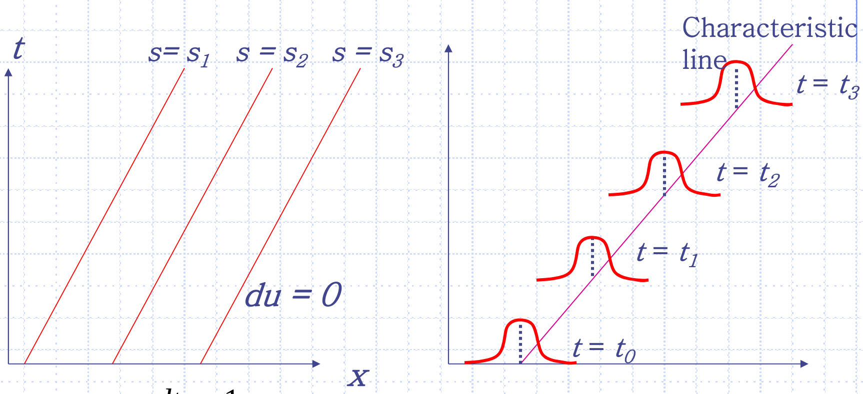
$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = f(x) \quad \text{I.C.} \end{cases}$$

- Application steps:
  - dx/dt =a =>$x=x_0+at$
  - du/dt =0 =>$u(t) = f(x_0)$
  - u(t)=f(x-at)



Characteristics of first-order PDEs

***u = f(s)* along the characteristic direction u =constant**



$$slope = \frac{dt}{dx} = \frac{1}{a}$$

Characteristics of first-order PDEs

- Along the characteristic direction:

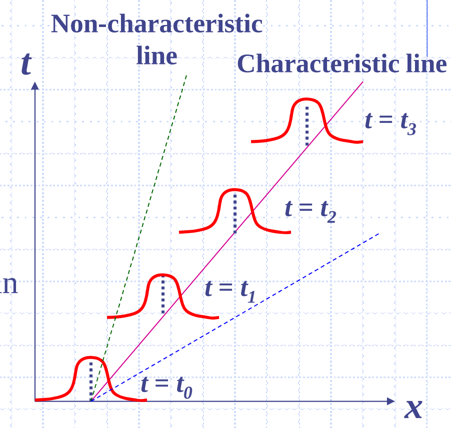
  $du = 0, \; u = \text{constant}$

  $u = f(s) = constant$

- The solution remains the same along the characteristic direction
- An observer moving with

  $s = \text{constant}$

  sees no changes the form of u
- The profile will change if the observer moves faster or slower than the characteristic line

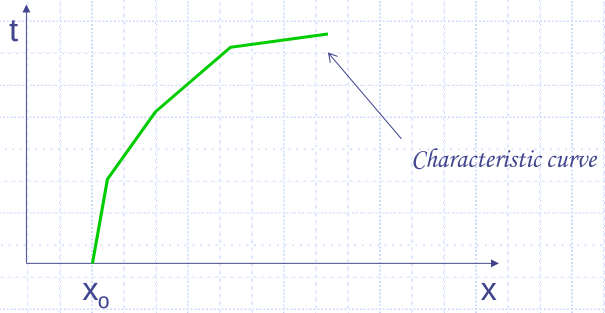


Case 2: Variable coefficient advection equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + a\dfrac{\partial u}{\partial x} = 0 \\ a = g(x,t) \\ IC: \quad u(x,0) = f(x) \end{cases}$$

❑ Application steps:

- dx/dt =a =>x=G(a)
- du/dt =0 =>u(t) =f(x_0)
- U(t)



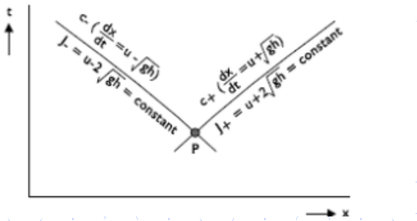
Case 2 Application: Saint Venant equations

- Characteristic form of Saint Venant eq

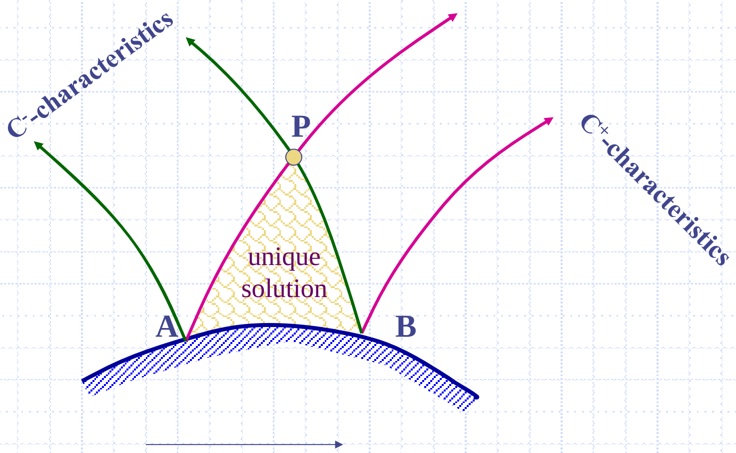
$$\begin{cases} \dfrac{\partial(u+2c)}{\partial t}+(u+c)\dfrac{\partial(u+2c)}{\partial x}=0 \\ \dfrac{\partial(u-2c)}{\partial t}+(u-c)\dfrac{\partial(u-2c)}{\partial x}=0 \end{cases}$$

$$\begin{cases} \dfrac{dx}{dt}=(u+c)\leftarrow C^{+} \\ \dfrac{dx}{dt}=(u-c)\leftarrow C^{-} \end{cases}$$

$$\begin{cases} J_{+}=(u+2c) \\ J_{-}=(u-2c) \end{cases}$$

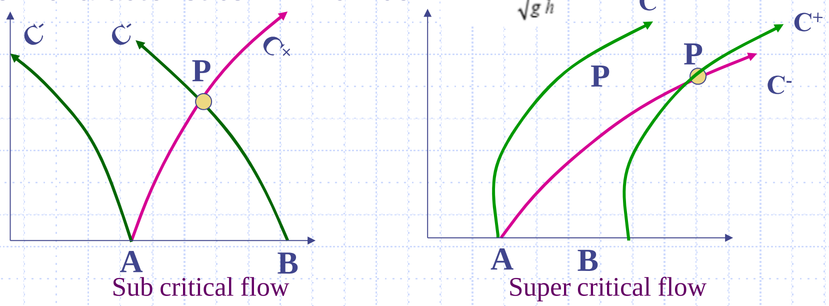


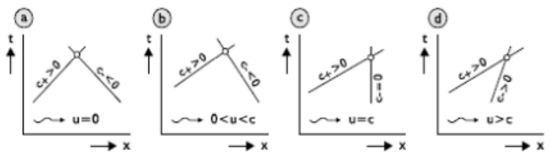
Characteristic directions



Characteristic directions

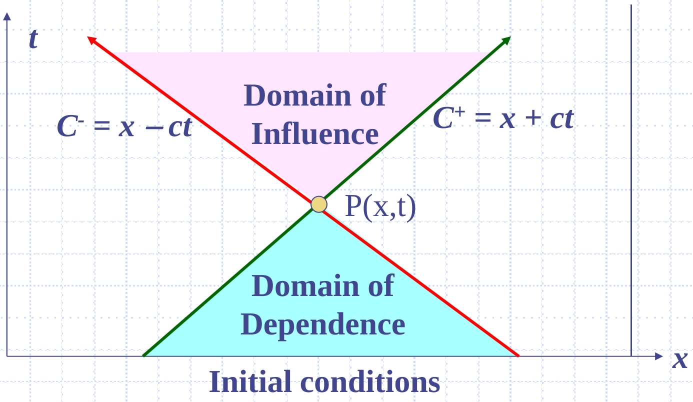
Flow characteristics: Fr number $Fr = \frac{|u|}{\sqrt{g\,h}}$





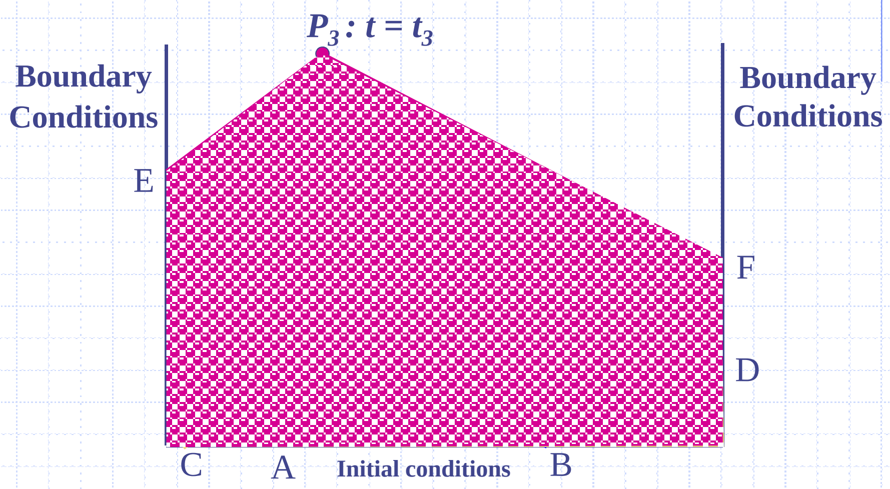
Physical Plane

Hyperbolic equation – propagation problem with no dissipation



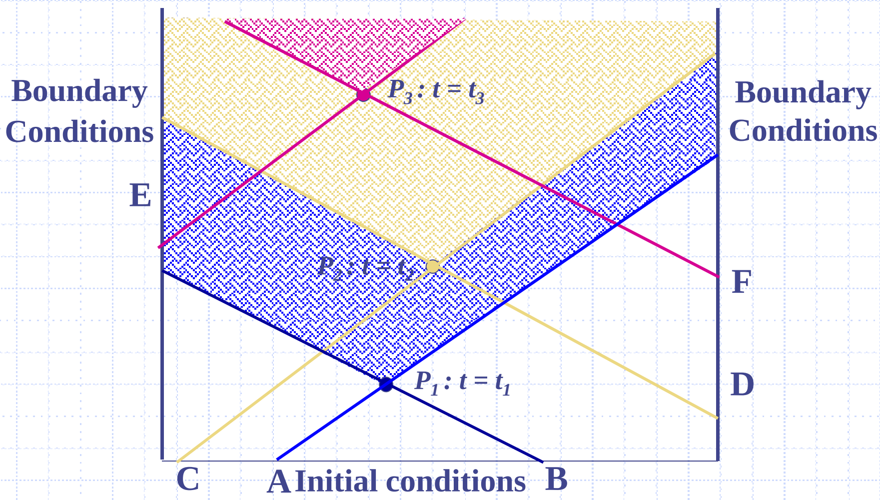
Characteristics - Propagation

Domain of Dependence

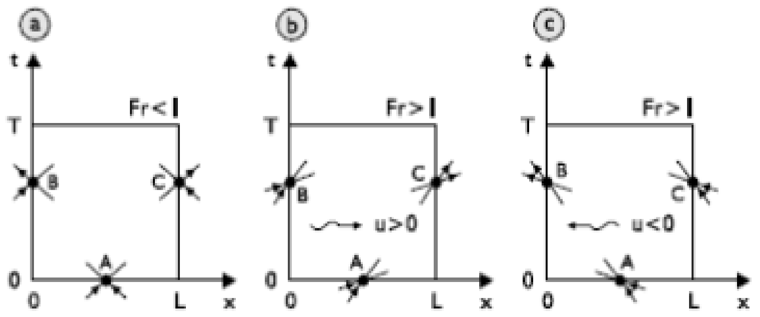


Characteristics - Propagation

Domain of Influence

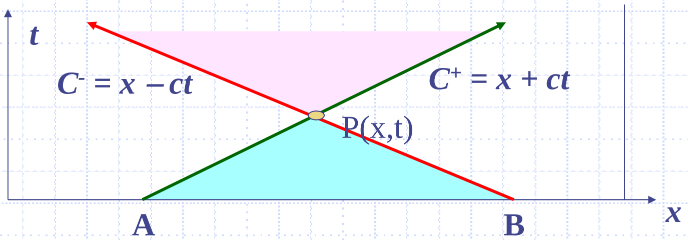


Characteristics: Boundary conditions



Summary

- The conditions derived along characteristics enable us to compute solutions from known conditions at an earlier point in time
- Example
  - If state of fluid in A is U=1m/s and h=5m, in B u=1.2m/s and h=4.8 m, compute u and h in P



(A)Hyperbolic PDEs

Finite difference method

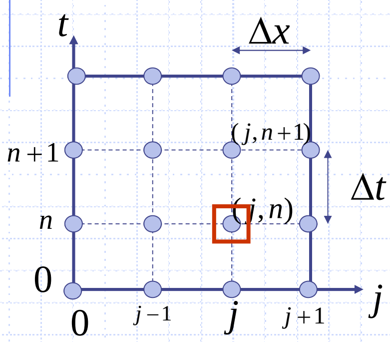
FDM to hyperbolic PDE

- Notations
  - F- forward
  - B- bacward
  - T-time
  - S –space
- Schemes
  - FTBS – explicit
  - BTBS – implicit
  - FTCS in n
  - CTCS in n

FDM-Hyperbolic PDE-Explicit schemes

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$

- FTBS



FT: $\frac{\partial U}{\partial t} \cong \frac{U_j^{n+1} - U_j^n}{\Delta t}$

BS: $\frac{\partial U}{\partial x} \cong \frac{U_j^n - U_{j-1}^n}{\Delta x}$

$Cr_j^n = \frac{a\Delta t}{\Delta x}$ *Courant number*

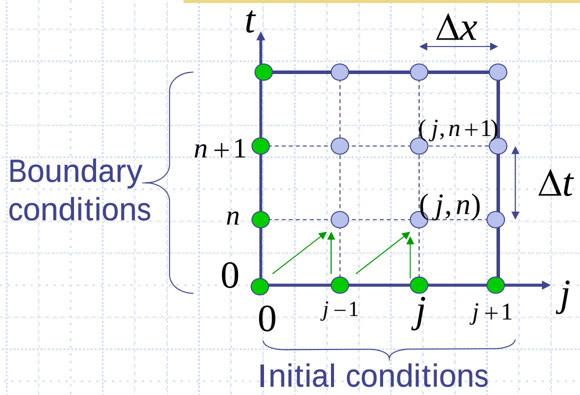
$$U_j^{n+1} = CrU_{j-1}^n + (1 - Cr)U_j^n$$

FDM-Hyperbolic PDE-Explicit schemes

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$

- FTBS- boundary and initial conditions

$$U_j^{n+1} = CrU_{j-1}^n + (1 - Cr)U_j^n$$

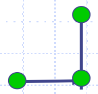


FDM-Hyperbolic PDE-Explicit schemes

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$

- FTBS- boundary and initial conditions

$$U_j^{n+1} = CrU_{j-1}^n + (1 - Cr)U_j^n$$



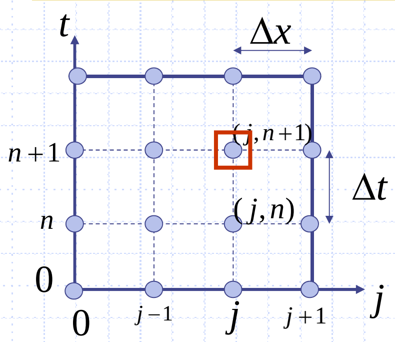
- For a given point, a *stencil* is a fixed subset of nearest neighbors
- A *stencil code* updates every point in a regular grid by "applying a stencil"

FDM-Hyperbolic PDE-Implicit schemes

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$

- BTBS

$$-CrU_{j-1}^{n+1} + (1 + Cr)U_j^{n+1} + (-Cr)U_j^n = 0$$



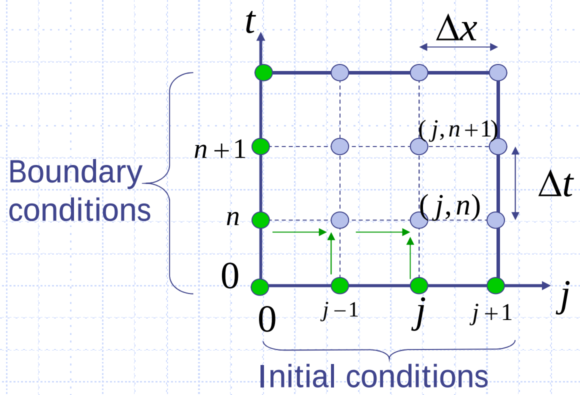
$$Cr_j^n = \frac{a\Delta t}{\Delta x}$$

FDM-Hyperbolic PDE-Implicit schemes

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$

❑BTBS- boundary and initial conditions

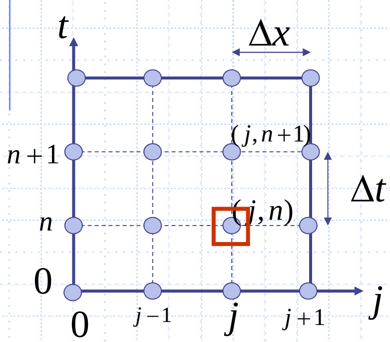
$$-CrU_{j-1}^{n+1} + (1+Cr)U_j^{n+1} + (-Cr)U_j^n = 0$$



FDM-Hyperbolic PDE – (Explicit)Schemes

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$

❑FTCS



FT: $\frac{U_j^{n+1} - U_j^n}{\Delta t}$ CS: $\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0$$

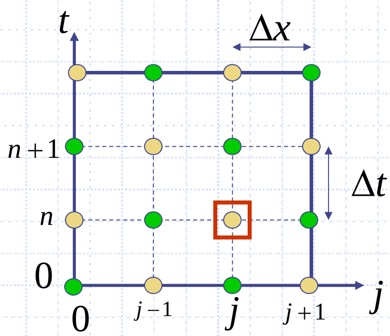
$$U_j^{n+1} = U_j^n - \frac{1}{2}Cr\left(U_{j+1}^n - U_{j-1}^n\right)$$

- unconditionally unstable
- used to compute very first time steps

FDM-Hyperbolic PDE-Upwind schemes

- Any BS scheme is also called *upwind schemes*, because information comes from the upstream
- Similarly the CS schemes are called *centered schemes*
- The CTCS scheme

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0$$



$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} + a\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} = 0$$

$$U_j^{n+1} = U_j^{n-1} - \frac{a\Delta t}{\Delta x}\left(U_{j+1}^n - U_{j-1}^n\right)$$

- two solutions for special BC

Other schemes

- Lax-Wendorf
- Cranck Nicholson

FD Schemes for hyperbolic system of equations

- Saint Venant equations: Unsteady, nearly horizontal flow

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q$$

$$\frac{\partial Q}{\partial t} + \frac{\partial \left(\alpha \frac{Q^2}{A} \right)}{\partial x} + gA \frac{\partial h}{\partial x} + \frac{gQ\,Q}{C^2 AR} = 0$$

Where:
- Q - discharge, $m^3 s^{-1}$
- A - flow area, m^2
- q - lateral flow, $m^2 s^{-1}$
- h - depth above datum, m
- C - Chezy resistance coefficient, $m^{1/2} s^{-1}$
- R - hydraulic radius, m
- α - momentum distribution coefficient

- Variables
  - two independent (x, t)
  - two dependent (Q, h)
- Conditions for solution
  - 2 point initial (Q, h)
  - 1 point up/downstream
    - h
    - Q
    - $Q=f(h)$

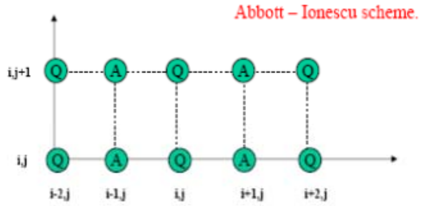
FD Schemes for hyperbolic system of equations

- Fully dynamic
- Diffusive wave – no inertia
- Kinematic wave- pure convective

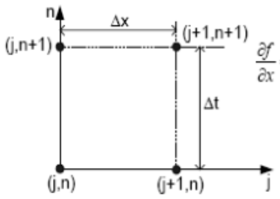
FD Schemes for hyperbolic system of equations

Example of schemes for full dynamic solution

Abbott – Ionescu scheme.



Four point Preissman scheme

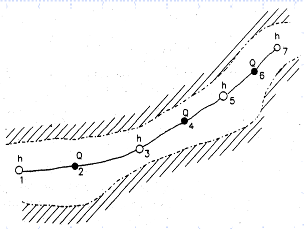


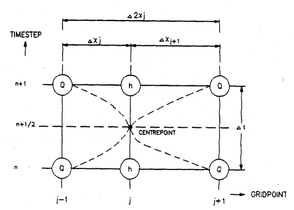
$$\frac{\partial f}{\partial x} = \frac{\theta\left(f_{j+1}^{n+1} - f_j^{n+1}\right) + (1-\theta)\left(f_{j+1}^{n} - f_j^{n}\right)}{\Delta x}$$

$$\frac{\partial f}{\partial t} = \frac{\left(f_j^{n+1} - f_j^{n}\right) + \left(f_{j+1}^{n+1} - f_{j+1}^{n}\right)}{2\Delta t}$$

Abbot – Ionescu scheme

- ❑Structured, cartesian grid
  - ❑ Implicit scheme (Abbott-Ionescu)
    - Continuity equation - h centered
    - Momentum equation - Q centered





Example discretization

Abbott Ionescu scheme

$$g\frac{\partial h}{\partial t} - u\frac{\partial u}{\partial t} + \left(gh-u^2\right)\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial h}{\partial t} \cong \frac{h_j^{n+1}-h_j^n}{\Delta t}$$

$$\frac{\partial u}{\partial t} \cong \frac{1}{2}\left(\frac{u_{j-1}^{n+1}-u_{j-1}^n}{\Delta t}+\frac{u_{j+1}^{n+1}-u_{j+1}^n}{\Delta t}\right)$$

$$\frac{\partial u}{\partial x} \cong \frac{1}{2}\left(\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x}+\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}\right)$$

$$h\frac{\partial u}{\partial t} - u\frac{\partial h}{\partial t} + \left(gh-u^2\right)\frac{\partial h}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} \cong \frac{u_{j+1}^{n+1}-u_{j+1}^n}{\Delta t}$$

$$\frac{\partial h}{\partial t} \cong \frac{1}{2}\left(\frac{u_j^{n+1}-u_j^n}{\Delta t}+\frac{u_{j+2}^{n+1}-u_{j+2}^n}{\Delta t}\right)$$

$$\frac{\partial h}{\partial x} \cong \frac{1}{2}\left(\frac{u_{j+2}^{n+1}-u_j^{n+1}}{2\Delta x}+\frac{u_{j+2}^n-u_j^n}{2\Delta x}\right)$$

Abbot – Ionescu scheme

- Transformation into linear equations

$$A1_j Q_{j-1}^{n+1} + B1_j h_j^{n+1} + C1_j Q_{j-1}^{n+1} = D1_j \quad \text{(mass)}$$

$$A1_j h_{j-1}^{n+1} + B1_j Q_j^{n+1} + C1_j h_{j-1}^{n+1} = D1_j \quad \text{(momentum)}$$

$$A1_j \phi_{j-1}^{n+1} + B1_j \phi_j^{n+1} + C1_j \phi_{j-1}^{n+1} = D1_j$$

- Tri-diagonal matrix form of equation

$$\begin{bmatrix} A_0\,B_0\,C_0 & & & & \text{all zeros} \\ & A_1\,B_1\,C_1 & & & \\ & & A_2\,B_2\,C_2 & & \\ & & & \cdots & \\ \text{all zeros} & & & \cdots & \\ & & & & A_{jj}\,B_{jj}\,C_{jj} \end{bmatrix} \cdot \begin{bmatrix} \phi_0 \\ \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \phi_{jj} \end{bmatrix}^{n+1} = \begin{bmatrix} D_0 \\ D_1 \\ D_2 \\ \cdot \\ \cdot \\ D_{jj} \end{bmatrix}^{n}$$

Abbot – Ionescu scheme

- ❑ Less equation than unknowns
- ❑ Use of suitable boundary conditions
- ❑ Introducing additional variables

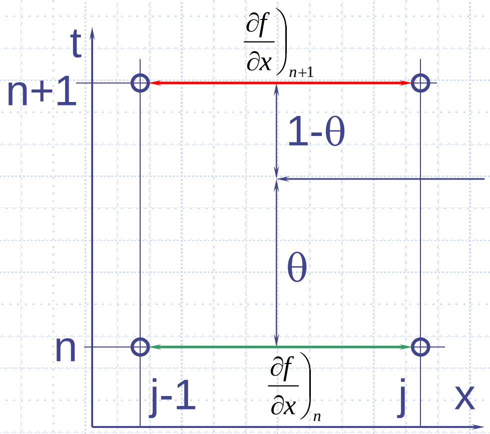
$$\phi_{j+1}^{n+1} = E_j \phi_j^{n+1} + F_j$$

- ❑ Substitution of into the linear equations
- ❑ Derivation of recurrence relations

Preismann scheme

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} = 0$$

4-points stencil (2 points in space, 2 levels in time)



f(x,t) can be either u, either h

$$\frac{\partial f}{\partial x} \cong \theta \left.\frac{\partial f}{\partial x}\right)_{n+1} + (1-\theta)\left.\frac{\partial f}{\partial x}\right)_{n}$$

with

$$\left.\frac{\partial f}{\partial x}\right)_{n} \cong \frac{f_j^n - f_{j-1}^n}{\Delta x}$$

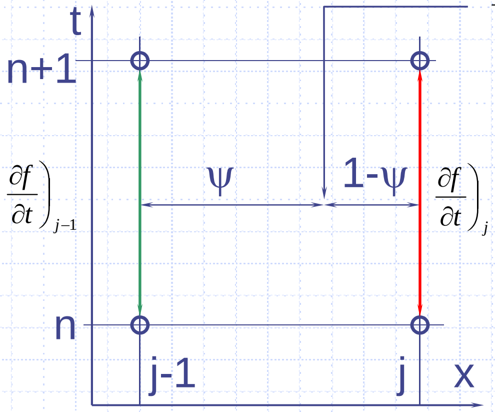
Preismann scheme

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} = 0$$

4-points stencil (2 points in space, 2 levels in time)

f(x,t) can be either u, either h

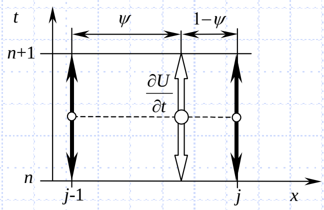
$$\frac{\partial f}{\partial t} \cong \psi \left.\frac{\partial f}{\partial t}\right)_j + (1-\psi)\left.\frac{\partial f}{\partial t}\right)_{j-1}$$



with

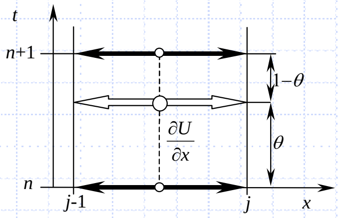
$$\left.\frac{\partial f}{\partial t}\right)_j \cong \frac{f_j^{n+1} - f_j^n}{\Delta x}$$

FDM-Hyperbolic PDE-Preismann scheme



$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial t} \cong \psi\frac{f_j^{n+1} - f_j^n}{\Delta t} + (1-\psi)\frac{f_{j-1}^{n+1} - f_{j-1}^n}{\Delta t}$$



$$\frac{\partial f}{\partial x} = \theta\frac{f_j^{n+1} - f_{j-1}^{n+1}}{\Delta x} + (1-\theta)\frac{f_j^n - f_{j-1}^n}{\Delta x}$$

θ = 0, the scheme is explicit,
θ = 1, the scheme is fully implicit

θ < 0.5 - fully unstable